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PART I: EXACT PHYSICAL PROPERTIES OF ELECTRON GASES IN UNIFORM MAGNETIC FIELDS AT T = 0 DEGREES K. PART II: THE PATH OF a SOUND WAVE IN GENERAL RELATIVITY.

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PART II:

THE PATH OF A SOUND WAVE IN GENERAL RELATIVITY

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by

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To my mother and wife

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TABLE OF CONTENTS

TABLE OF CONTENTS (cont'd)

LIST OF FIGURES

LIST OF FIGURES (cont'd)

LIST OF FIGURES (cont'd)

LIST OF FIGURES (cont'd)

ABSTRACT

PART I

An exact analysis is presented to obtain the physical properties of free Schrödinger and Dirac electron gases in uniform magnetic fields of arbitrary strength at $T = 0^{\circ}$ K. The newly defined functions, $\Lambda_{\mu}(s) = \sum_{0}^{[s]} (s-n)^{\mu}$, for $\mu = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, are employed to obtain the total energies, pressures, Fermi energies, magnetizations, susceptibilities, and their singularities as functions of B. In both theories, Schrödinger and Dirac, it is shown that the spin susceptibility χ_s may be decomposed into two parts, χ_{s1} and χ_{s2}, where χ_{s2} is oscillatory and purely diamagnetic in all field strength B. A graphical method of finding the Fermi energy $\varepsilon_F(B)$ for a given value of B is obtained. The system is shown to become an effectively one-dimensional electron gas in a field B greater than $B_{\downarrow} = (2\eta n)^{2/3}$, where $\eta = 1.181 \times 10^{-10}$ CGS and n is the electronic density. The Schrödinger case is extended to the Bloch electrons on the ellipsoidal constant energy surface. The Dirac case is applied to a white dwarf in an external magnetic field and there we prove that white dwarfs can not be formed in an external magnetic field if $B \geq (2\eta n_o)^{2/3}$, where n_o is the central number density of the itinerant electrons.

PART II

The general path equation of interior sound waves and exterior light waves is derived. It is shown explicitly that the sound path deflects towards stronger gravitational fields and is expected to deflect less in the medium with polarization than without it using Einstein-Cartan theory of gravitation.

PART I

EXACT PHYSICAL PROPERTIES OF ELECTRON GASES IN UNIFORM MAGNETIC FIELDS AT $T = 0^{\circ}$ K

CHAPTER I

INTRODUCTION

The purpose of Part I is two-fold:

1. To obtain in a unified way the exact formulations of the physical properties of the non-interacting Schrödinger and Dirac electron gases at $T = 0^{\circ}$ K for an arbitrary strength of the magnetic field B. We are primarily interested in the magnetic properties.

2. To show explicitly the nature of the approximations inherent in the existing formulae.

The magnetic properties of the free electron gas in a uniform magnetic field were studied by various authors since Landau[1,2] obtained the energy eigenvalues and states in 1930. Peierls[3] gave the first theoretical interpretation of the de Haas-van Alphen effect, although his mathematical discussion was not complete. Later, Sondheimer et al.[4] and Wilson[5] derived fairly accurate formulations valid at low temperatures and in the weak field limit. However, it should be noticed that these are in no way rigorously exact at both non-zero and zero temperatures. Here it is our main purpose and interest to obtain the exact formulae at $T = 0^{\circ}$ K.

The crucial tool for our exact analysis is the newly defined function $\Lambda_{\mu}(s)$ discussed in Appendix A. This function enables us to handle the oscillatory singularities in the density of states and various cusps occurring in the thermodynamic functions due to the nature of the $T = 0$ case. These are the main features in the zero-temperature formalisms. We find the exact dependence of the Fermi energy on B and this is taken into account when the total energy is differentiated with respect to B to derive the magnetization. Here we point out that all the existing formulas neglect this part.[5,6] Consequently these formulas contain the oscillatory terms $\sum_1^\infty \frac{\sin(2\pi n s_F - \pi/4)}{n^{3/2}}$, whereas the present theory contains the term $\sum_1^\infty \frac{\cos(2\pi n S_F - \pi/4)}{n^{5/2}}$, instead (see Eqs. (2.14') and (2.14")). The cusps in the function $\Lambda_{1/2}(s)$ allows us to determine the exact formulas for the field strength B_K corresponding to the (K+1)'th Landau level, as in Eq. (2.4), whereas the singularities in the function $\Lambda_{-1/2}(s)$ offers a way to derive the discontinuities occurring in the susceptibilities defined by $\chi = \frac{\partial M}{\partial B}$. Our unified way of dealing with an arbitrary field strength also leads to the ultra-strong field case in which all electrons occupy the 1st Landau level and the system becomes an effectively one-dimensional electron gas. We also prove that the power series expansions of the thermodynamic quantities, as functions of the field strength B, do not exist due to the singular factor $\chi_t(0)$. In Chapter III we investigate the magnetic properties of the Dirac electron gas using similar techniques as in the Schrödinger gas. There we transform the E-spectrum into the T-spectrum. The resulting formulas resemble the Schrödinger case, namely $T = \frac{E}{2mc^2}(E+2mc^2)$. In Dirac theory the function $\Sigma_{-1/2}(a;s) = \int_0^s \sqrt{1+at}\, \Lambda_{-1/2}(t)\, dt$ plays the role comparable to the function $\Lambda_\mu(s)$ in the Schrödinger theory. We discuss the ultra-strong and zero field limits as in the Schrödinger case. Finally, we take the NR (non-relativistic) limit and insure that all the expressions reduce to those of the Schrödinger theory. The Dirac theory is then applied to a white dwarf in an external magnetic field and we prove a theorem stating that white dwarfs can not be formed in an external magnetic field B if $B \geq (2\eta n_o)^{2/3}$, where n_o is the central electronic density.

Finally, the present technique is applied to a Bloch electron gas in opposing electric and magnetic fields at $T = 0^\circ$ K and we obtain the surface magnetic properties in the weak magnetic field limit. There we show that susceptibilities do not possess discontinuities even at $T = 0^\circ$ K. This is due to the nature of the bound states and the non-existence of free particle behavior, unlike the case of a system with magnetic field alone.

CHAPTER II

THE SCHRÖDINGER ELECTRON GAS

We consider here a non-interacting electron gas in a magnetic field $\underset{\sim}{B}$ along the z-axis. The normalized wave function of an electron is[1]

$$\psi_{n,k_y,k_z}(\underset{\sim}{X}) = \left[\frac{\alpha}{\pi^{1/2}\, 2^n\, n!\, L_2 L_3} \right]^{1/2} H_n[\alpha(x+x_o)]$$

$$e^{-\frac{1}{2}\alpha^2(x+x_o)^2} \exp(ik_y y + ik_z z) \ ,$$

where the gauge $\underset{\sim}{A} = (0,Bx,0)$ is used and H_n is the Hermite polynomial of order n, $\alpha^2 = m\omega\, c/\hbar$, $x_o = c\hbar k_y/eB$, and $\omega_c = eB/mc$, the cyclotron frequency. We use periodic boundary conditions along the y and z directions in the crystal volume $V = L_1L_2L_3$. Then k_y and k_z are quantized as follows.

$$k_y = \frac{2\pi}{L_2} n_y,\ n_y = 0,\pm1,\pm2 \ \ldots,\ k_z = \frac{2\pi}{L_3} n_z,\ n_z = 0,\pm1,\pm2 \ \ldots$$

The energy eigenvalues are

$$E_n(k_z,\sigma) = (n + \tfrac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m} + \sigma\mu_o B \ , \qquad (2.1)$$

which has the degeneracy $g = 2eBL_1L_2/hc$. Here μ_o is the

Bohr magneton $e\hbar/2mc$, and $\sigma = \pm 1$ for spin-up or spin-down. Now we notice that the total number of quantum states below or equal to the energy value E will be

$$N(E) = \frac{2eBV(2m)^{1/2}}{ch^2} \sum_{0}^{[\,]} \{E-(n+\tfrac{1}{2})\,\hbar\omega_c\}^{1/2} ,$$

in which [] should read $[\frac{E}{\hbar\omega_c} - \frac{1}{2}]$, using the Gauss symbol. This is easily derived by noticing that the range of χ_o is $-L_1/2 < x_o \leq L_1/2$. The density of states then becomes

$$D(E) = \frac{dN}{dE} = \frac{eB(2m)^{1/2}V}{ch^2} \sum_{0}^{[\,]} \{E-(n+\tfrac{1}{2})\hbar\omega_c\}^{-1/2} .$$

Introducing a new dimensionless parameter s defined by $s = \frac{E}{\hbar\omega_c} - \frac{1}{2}$, we finally arrive at

$$N(E) = 2\hbar\omega_c\, A\, \Lambda_{1/2}(s) \quad \text{and} \quad D(E) = A\Lambda_{-1/2}(s), \qquad (2.2)$$

where

$$A = \frac{eBV}{h^2c} \left(\frac{2m}{\hbar\omega_c}\right)^{1/2} .$$

A. Exact Fermi Energy $\varepsilon_F(B)$

We now observe that the number of spin-up electrons is given by

$$N_\uparrow = \int_{\mu_oB+\hbar\omega_c/2}^{\varepsilon_F} D(E-\mu_oB)\,dE = A\hbar\omega_c \int_{0}^{s_F-1} \Lambda_{-1/2}(s)\,ds .$$

For spin-down electrons we have

$$N_{\downarrow} = \int_{-\mu_0 B+\hbar\omega_c/2}^{\varepsilon_F} D(E+\mu_0 B)\,dE = A\hbar\omega_c \int_0^{s_F} \Lambda_{-1/2}(s)\,ds \ ,$$

where $s_F = \varepsilon_F/\hbar\omega_c$. Then the condition $N = N_{\uparrow}+N_{\downarrow}$ gives the equation

$$\Lambda_{1/2}(s_F) = \frac{1}{2}\, s_F^{1/2} + \eta n B^{-3/2} \ , \tag{2.3}$$

where Eqs. (A.1.2) have been used. Here, $\eta = (\pi^4 c^3 \hbar^3/2e^3)^{1/2} = 1.181\mathrm{x}10^{-10}$ in cgs units, and n is the electronic density. Eq. (2.3) can be solved graphically to yield s_F and therefore $\varepsilon_F (=\hbar\omega_c s_F)$ for given values of n and B. This is illustrated in Fig. 1.

We now define B_k as the field strength corresponding to $s_F = k+1$. In other words, if $B_k < B \leq B_{k-1}$, then $k \leq s_F < k+1$; therefore, we readily obtain the following by inspecting Figs. 1 and 2:

$$B_0 < B < \infty \ , \quad 0 < s_F < 1 \ , \quad B_0 = (2\eta n)^{2/3} \ ;$$

$$B_1 < B \leq B_0 \ , \quad 1 \leq s_F < 2 \ , \quad B_1 = \left(\frac{2\eta n}{2+2^{1/2}}\right)^{2/3} \ ;$$

$$\ldots\ldots \ ; \tag{2.4}$$

$$B_k < B \leq B_{k-1} \ , \qquad k \leq s_F < k+1 \ ,$$

$$B_k = \left[\frac{2\eta n}{2+2(2)^{1/2}+\ldots+2k^{1/2}+(k+1)^{1/2}} \right]^{2/3} ;$$

Furthermore we notice that

(i) if $0 < s_F < I$, then $\Lambda_{1/2}(s_F) = s_F^{1/2}$ and Eq. (2.3) gives $\varepsilon_F(B) = 8\eta^2 n^2 \mu_o / B^2$;

(ii) if $I \leq s_F < 2$, then $\Lambda_{1/2}(s_F) = (s_F)^{1/2} + (s_F-1)^{1/2}$ and we obtain

$$\{8s_F^2 - \frac{31}{4} - (\eta n B^{-3/2})^2 + 2\eta n B^{-3/2}(2s_F-1)+1\}^2$$

$$= 4\ s_F^2(s_F-1) \ , \qquad (2.5)$$

which yields a unique solution in this interval of s_F. In principle, this method can be extended to yield the equations for s_F of higher values.

Figure 2 shows s_F as a function of B for $n = 10^{22}\text{cm}^{-3}$. s_F has cusps at $s_F=k$, $k=1.2\ldots$, due to cusps of the function $\Lambda_{1/2}(s_F)$ at the same points. The Fermi energy $\varepsilon_F(B)$ as a function of B is shown in Fig. 3, and Fig. 4 shows this Fermi energy as a function of s_F parameter, which shows diminishing ripples as B decreases and cusps at $B=B_{k-1}$ or $s_F=k$.

The occurrence of cusps in the functions $s_F(B)$ and $\mathcal{E}_F(B)$ is easily seen as follows. Differentiating Eq. (2.3) with respect to B we obtain

$$\frac{ds_F}{dB} = \frac{6\eta\, nB^{-5/2}\, s_F^{1/2}}{1-2s_F^{1/2}\Lambda_{-1/2}(s_F)} \qquad (\leq 0)\ . \tag{2.6}$$

This function is singular due to the presence of the singular function $\Lambda_{-1/2}(s_F)$ in the denominator. Therefore the function $s_F(B)$ is continuous but has cusps at $s_F=k$. A similar argument applies to the function $\varepsilon_F(B)$. We first observe that

$$\frac{d\varepsilon_F(B)}{dB} = \hbar\omega_c\ \left(\frac{s_F}{B} + \frac{ds_F}{dB}\ \right)$$

which is a singular function. For similar reasons $\mathcal{E}_F(B)$ is continuous but has cusps at $s_F=k$, where $k = 1,2,\ldots$.

Before we discuss the susceptibilities in the next section, we note the formulas

$$\Delta\left(\frac{1}{B_k}\right)^{3/2} \equiv \left(\frac{1}{B_k}\right)^{3/2} - \left(\frac{1}{B_{k-1}}\right)^{3/2} = \frac{k^{1/2}+(k+1)^{1/2}}{2\eta n}$$

or equivalently

$$\Delta(\frac{1}{B_k}) \approx \frac{k^{1/2}+(k+1)^{1/2}}{3\eta n} B_k^{1/2} . \quad (2.7)$$

However, for large k, $k^{1/2}+(k+1)^{1/2} \approx 2(k+\frac{1}{2})^{1/2}$. Then setting $s_F = k + \frac{1}{2}$ and dropping the index k we also arrive at $\Delta(\frac{1}{B}) = (\frac{3}{2}\eta n)^{-2/3}$ for the small B, or large s_F limit. Formula (2.18) given in Section D has been used in this limit.

B. Susceptibilities

From the previous expressions of $N_\uparrow$ and $N_\downarrow$ we obtain the magnetization due to spins as

$$M_s = \mu_o(N_\downarrow - N_\uparrow) = A\hbar\omega_c\, \mu_o \int_{s_F-1}^{s_F} \Lambda_{-1/2}(s)\,ds$$

$$= \frac{\mu_o N}{2\eta n} B^{3/2} s_F^{1/2} , \quad (2.8)$$

where the shift property of $\Lambda_{1/2}$, Eq. (A.1) has been used. The spin susceptibility is then given by

$$\chi_s = \frac{\partial M_s}{\partial B} = \frac{3\mu_o N}{4\eta n} (Bs_F)^{1/2} + \frac{\mu_o N}{4\eta n} \frac{B^{3/2}}{s_F^{1/2}} (\frac{ds_F}{dB}) . \quad (2.9)$$

Using Eq. (2.6) we finally obtain

$$\chi_s = \frac{3\mu_o N}{4\eta n}(Bs_F)^{1/2} - \frac{3}{2}\frac{\mu_o N}{B}\frac{1}{2s_F^{1/2}\Lambda_{-1/2}(s_F)-1} = \chi_{s1}+\chi_{s2}. \tag{2.9'}$$

It is easily seen that the oscillatory nature of χ_s (ultimately χ_t as shown later) is entirely due to the factor ds_F/dB in χ_{s2} since this term contains the singular function $\Lambda_{-1/2}(s_F)$, the density of states. The oscillating factor $\{2s_F^{1/2}\Lambda_{-1/2}(s_F)-1\}^{-1}$ in ds_F/dB is shown in Fig. 5. Here χ_{s1} and $-\chi_{s2}$ are always positive in all field strengths B; therefore, we see that χ_s may be decomposed into two parts, χ_{s1} and χ_{s2}. Later it is shown that $\chi_{s2} = -\chi_{s1}$, $\chi_s = 0$ for $0<s_F\leq 1$ and

$$\chi_{s1} \underset{B=0}{\sim} \frac{3}{2}\chi_p, \langle\chi_{s2}\rangle \underset{B=0}{\sim} -\frac{1}{2}\chi_p, \langle\chi_s\rangle \underset{B=0}{\sim} \chi_p ,$$

and

$$\chi_{s2} \underset{B\geq B_\downarrow}{\sim} -\chi_{s1} ,$$

where χ_p is the Pauli spin susceptibility given by $\chi_p = \frac{3}{4}\mu_o N(\frac{3}{2}\eta n)^{-2/3}$ and the symbol <> is defined in Section D. Using the formulas (A.4) and (2.4) we obtain the discontinuities in $\chi_s(s_F)$ of Eq. (2.9) as

$$\chi_s(k_+)-\chi_s(k_-) = \frac{3\mu_o N[1+2^{1/2}+\ldots+(k-1)^{1/2}+\frac{1}{2}k^{1/2}]^{2/3}}{4(\eta n)^{2/3}k^{1/2}[1+2^{-1/2}+\ldots+(k-1)^{-1/2}+\frac{1}{2}k^{-1/2}]}\ , \tag{2.10}$$

where $\chi_s(k_{\pm}) = \lim_{\varepsilon\to 0}\chi_s(k\pm\varepsilon)$ and k is a positive integer. Fig. 6 shows χ_s, χ_{s1}, and χ_{s2} as functions of s_F for $n = 10^{22}\ cm^{-3}$ in units of λ_p. Next we consider the total energy, magnetization, and susceptibility.

The total energy E_t of the non-interacting electron gas at $T = 0^oK$ may be written as $E_t = E_{\uparrow}+E_{\downarrow}$, where

$$E_{\uparrow} = \int_{\mu_o B+\hbar\omega_c/2}^{\varepsilon_F} ED(E-\mu_o B)\,dE = A(\hbar\omega_c)^2\int_0^{s_F-1}(s+1)$$

$$\Lambda_{-1/2}(s)\,ds$$

and

$$E_{\downarrow} = \int_{\mu_o B+\hbar\omega_c/2}^{\varepsilon_F} ED(E+\mu_o B)\,dE = A(\hbar\omega_c)^2\int_0^{s_F}\Lambda_{-1/2}(s)\,ds\ .$$

This gives the total energy as

$$E_t = \frac{\mu_o N}{\eta n}B^{5/2}\ F(s_F)\ , \tag{2.11}$$

where $F(s_F)$ is given by $F(s_F) = 2s_F\Lambda_{1/2}(s_F) - \frac{4}{3}\Lambda_{3/2}(s_F) - \frac{1}{3}s_F^{3/2}$. The pressure is then given by

$$P(B) = -\frac{\partial E_t}{\partial V} = \frac{N}{V^2}\frac{\partial E_t}{\partial n} = \frac{2\mu_o}{3\eta} B^{5/2} \{2\Lambda_{3/2}(s_F) - s_F^{3/2}\} , \tag{2.12}$$

where the formula

$$\frac{ds_F}{dn} = \frac{4\eta B^{-3/2}}{2\Lambda_{-1/2} - s_F^{-1/2}} \tag{2.13}$$

derived from Eq. (2.3) has been used. The total magnetization M_t is obtained in terms of $F(s_F)$ as

$$M_t = -\frac{\partial E_t}{\partial B} = -\frac{5}{2}\frac{\mu_o N}{\eta n} B^{3/2} F(s_F) - \frac{\mu_o N}{\eta n} B^{5/2} \frac{dF}{dB} .$$

However, it is easily observed that

$$\frac{dF}{dB} \left(= \frac{dF}{ds_F}\frac{ds_F}{dB}\right) = -3\eta n \, s_F \, B^{-5/2} ,$$

which finally gives M_t and $\chi_t (=\partial M_t/\partial B)$ as

$$M_t = \frac{5}{3}\frac{\mu_o N}{\eta n} B^{3/2} \{2\Lambda_{3/2}(s_F) - s_F^{3/2}\} - 2\mu_o N \, s_F , \tag{2.14}$$

$$\chi_t = 3\mu_o N \frac{ds_F}{dB} + \frac{5}{2}\frac{\mu_o N}{\eta n} B^{1/2} \{2\Lambda_{3/2}(s_F) - s_F^{3/2}\} . \tag{2.15}$$

It is not difficult to realize that the oscillatory nature of χ_t is again entirely from the factor ds_F/dB in the first term of Eq. (2.15). Next we derive an alternative representation of the function $\Lambda_{3/2}(s)$ by obtaining E_t using the method of Sondheimer and Wilson.[4] The total energy E_t may be written as[6]

$$E_t = N\varepsilon_F - Z(\varepsilon_F) \ ,$$

where

$$Z(\varepsilon_F) = \hbar\omega_c V \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \left(\frac{1}{2\pi i}\right) \int_{c-i}^{c+i} \beta^{-5/2} e^{E\beta} \coth \frac{\beta\hbar\omega_c}{2} \, d\beta$$

and where the contour of integration is shown in Ref. 6. By means of the formulae

$$\coth Z = \sum_{p=0}^{\infty} \frac{2^{2P} B_{2p}}{(2P)!} Z^{2P-1}$$

and

$$\frac{1}{2\pi i} \int_{-C_1} Z^{-\nu} e^{Zt} \, dZ = t^{\nu-1}/\Gamma(\nu) \ ,$$ [6]

we finally arrive at

$$E_t = N\varepsilon_F - V \frac{16(\hbar\omega_c)^{5/2}}{15\sqrt{\pi}} \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \left[s_F^{5/2} + \frac{5}{16}\sqrt{s_F} + \frac{15\sqrt{\pi}}{8} \sum_{P=2}^{\infty} \frac{B_{2P}}{(2P)!\,\Gamma(7/2 - 2P)\, s_F^{2P-5/2}}\right.$$

$$- \frac{15}{16\pi^2\sqrt{2}} \sum_{1}^{\infty} \frac{\cos(2\pi n s_F - \pi/4)}{n^{5/2}} \quad] \, , \tag{2.11'}$$

where B_n is the Bernoulli number. Eq. (2.11') reduces to the existing formulas[5,6] when the third term in the [ ] is neglected in the limit of $s_F \gg 1$ (or $\varepsilon_F \gg \hbar\omega_c$), namely the weak field limit.

Here we point out that the existing formulas[5,6] for non-zero temperatures, namely the free enrgy F and the total number of particles N $(= \frac{\partial F}{\partial \mu})$, involve the approximations $\hbar\omega_c \ll \mu$ and $KT \ll \mu$. The first approximation is employed when the hyperbolic function is expanded[6] in a power series whereas the second one is due to the replacement of $-\frac{\mu}{KT}$ by $-\infty$. For the case of M_t $(= -\frac{\partial F}{\partial B})$ one further approximation is involved, $\frac{\partial \mu}{\partial B} = 0$. This will be demonstrated explicitly in the later discussion of this quantity. Comparing Eq. (2.11) with Eq. (2.11') yields

$$\Lambda_{3/2}(s) = \frac{2}{5} S^{5/2} + \frac{1}{2} S^{3/2} + \frac{1}{8} S^{1/2} + \frac{3\sqrt{\pi}}{4}$$

$$\sum_{P=2}^{\infty} \frac{B_{2P}}{(2P)!\Gamma(7/2-2P) S^{2P-5/2}} - \frac{3}{8\sqrt{2}\pi^2} \sum_{1}^{\infty} \frac{\cos(2\pi n S - \pi/4)}{n^{5/2}} . \tag{2.16}$$

Now we demonstrate that Eq. (2.16) is an approximation generally valid for $S \gg 1$. Using the Poisson summation formula[7]

$$\sum_{n_1}^{n_2} f(n+\tfrac{1}{2}) = \sum_{-\infty}^{\infty} (-)^{P} \int_{n_1}^{n_2+1} f(x)\, e^{2\pi i P}\; dx$$

we obtain

$$\Lambda_{3/2}(s) = \sum_{-\infty}^{\infty} (-)^{n}\, e^{2\pi i n(s+1/2)} \int_{\langle s\rangle-1/2}^{s+1/2} t^{3/2}\, e^{-2\pi i n t}\, dt = \sum_{-\infty}^{\infty} (-)^{n}\, e^{2\pi i n(s+1/2)} I(n,s)\;,$$

where

$$I(n,s) = \begin{cases} n=0; & \frac{2}{5}(s+\frac{1}{2})^{5/2} - \frac{2}{5}\left|\langle s\rangle-\frac{1}{2}\right|^{5/2} \\ n\neq 0; & e^{-2\pi i n(s+1/2)}\left[\dfrac{3\sqrt{s+\frac{1}{2}}}{8\pi^2 n^2} - \dfrac{(s+\frac{1}{2})^{3/2}}{2\pi i n}\right] \end{cases}$$

$$- \frac{3}{16\pi^2 n^2}(2in)^{-1/2}\,\mathrm{erf}\left(\sqrt{2\pi i n(s+\tfrac{1}{2})}\right)$$

$$- \begin{cases} e^{-2\pi i n(\langle s\rangle-\frac{1}{2})}\left[\dfrac{3\sqrt{\langle s\rangle-\frac{1}{2}}}{2\pi i n} - \dfrac{(\langle s\rangle-\frac{1}{2})^{3/2}}{2\pi i n}\right] \\ \quad + \dfrac{3}{16\pi^2 n^2}(2in)^{1/2}\,\mathrm{erf}\left(\sqrt{2\pi i n(\langle s\rangle-\frac{1}{2})}\right) & \text{if } \langle s\rangle > \frac{1}{2} \\ \hline i\left[e^{2\pi i n(\frac{1}{2}-\langle s\rangle)}\left[\dfrac{3\sqrt{\frac{1}{2}-\langle s\rangle}}{8\pi^2 n^2} + \dfrac{(\frac{1}{2}-\langle s\rangle)^{3/2}}{2\pi i n}\right]\right. \\ \quad - \dfrac{3}{16\pi^2 n^2}(-2in)^{-1/2}\,\mathrm{erf}\left(\sqrt{-2\pi i n(\frac{1}{2}-\langle s\rangle)}\right) & \text{if } \langle s\rangle < \frac{1}{2} \end{cases}$$

and

$$\mathrm{erf}(Z) \equiv \frac{2}{\sqrt{\pi}} \int_0^Z e^{-t^2}\, dt \ ,$$

the error function. Here we used the formula

$$\int_0^{\varepsilon} x^{3/2}\, e^{-2\pi i P x}\, dx = \frac{2}{5}\, \varepsilon^{5/2} \delta_{po} \ + e^{-2\pi i P \varepsilon}$$

$$\left[\frac{3\sqrt{\varepsilon}}{8\pi^2 P^2} - \frac{\varepsilon^{3/2}}{2\pi i P} \right] - \frac{3}{16\pi^2 P^2} (2iP)^{-1/2}\, \mathrm{erf}(\sqrt{2\pi i P \varepsilon}) \ .$$

If $s \gg 1$ the terms involving $|\langle s\rangle - \frac{1}{2}|$ may be neglected. Then we use the asymptotic expansion of the error function $\mathrm{erf}(Z)$, namely

$$\mathrm{erf}(Z) \underset{|Z|\,\mathrm{large}}{\sim} 1 - \frac{2e^{-Z^2}}{Z\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{(-)^{\nu}(2\nu-1)!}{(4Z^2)^{\nu}(\nu-1)!} \ .$$

Finally we arrive at

$$\Lambda_{3/2}(s) \approx \frac{2}{5}\,(s+\tfrac{1}{2})^{5/2} - \frac{1}{16}\,(s+\tfrac{1}{2})^{1/2} - \frac{3}{8} \sum_{1}^{\infty}$$

$$\frac{B_{2k}[2^{4k-1}-1][8k-7]!}{8^{4k-3}(s+\frac{1}{2})^{4k-5/2}(4k-4)!(4k)!}$$

$$- \frac{3}{8\pi^2\sqrt{2}} \sum_{1}^{\infty} \frac{\cos(2\pi n s - \pi/4)}{n^{5/2}}$$

$$= \frac{2}{5}\, s^{5/2} + \frac{1}{2}\, s^{3/2} + \frac{1}{8}\, s^{1/2} + \frac{1}{64.30.s^{3/2}}$$

$$+ \ldots - \frac{3}{8\pi^2\sqrt{2}} \sum_1^{\infty} \frac{\cos(2\pi ns - \pi/4)}{n^{5/2}}$$

This is the expression obtained by Dingle.[8] Now returning to our previous expansion Eq. (2.16), we notice that this also gives the same leading terms as the above expansion. Consequently we realize that the method leading to Eq. (2.16) does not render the exact expression of $\Lambda_{3/2}(s)$ since we assumed $\beta = \frac{1}{kT}$ is finite rather than infinite during the derivation. Therefore Eq. (2.16) is valid for $s >> 1$. Here we adopt the expansion Eq. (2.16) for further discussions.

By the use of Eq. (2.16) the earlier M_t is now expressed as

$$M_t \simeq \frac{5\mu_o N}{3\eta\, n} B^{3/2} \left\{ \frac{4}{5}\, s_F^{5/2} + \frac{1}{4}\, s_F^{1/2} + \frac{3\sqrt{\pi}}{2} \sum_{P=2}^{\infty} \right.$$

$$\frac{B_{2P}}{(2P)!\,\Gamma(7/2-2P)\, s_F^{2P-5/2}} - \frac{3}{4\sqrt{2}\,\pi^2} \sum_1^{\infty}$$

$$\left. \frac{\cos(2\pi n s_F - \pi/4)}{n^{5/2}} \right\} - 2\mu_o N s_F \qquad (2.14')$$

However, Eq. (2.14') can not be compared with the existing formulas in the $T = 0^{o}K$ limit since the latter did not take account of the B dependence of ϵ_F when the free energy was differentiated with respect to B in arriving at M_t.[5,6] Nevertheless we can obtain the approximate expression for M_t and show that this reduces to the existing formula at $T = 0^{o}K$ as follows. Using the formula Eq. (2.21),

$$s_F \underset{B=0}{\sim} \frac{(\frac{3}{2}\eta n)^{2/3}}{B} = \frac{k}{B}$$

as derived in Section D, we write Eq. (2.11') as

$$E_t \simeq 2N\mu_o K - \frac{V16(2\mu_o)^{5/2}B^{5/2}}{5\sqrt{\pi}} \left(\frac{m}{2\pi\hbar^2}\right)^{3/2}$$

$$\left[\left(\frac{K}{B}\right)^{5/2} + \frac{5}{16}\sqrt{\frac{K}{B}} - \frac{15}{16\pi^2\sqrt{2}} \sum_1^\infty \frac{\cos(2\pi nk/B-\pi/4)}{n^{5/2}}\right] .$$

Now $M_t = -\partial E_t/\partial B$ becomes

$$M_t = \frac{16V(2\mu_o)^{5/2}B^{3/2}m^{3/2}}{15\sqrt{\pi}(2\pi\hbar^2)^{3/2}} \left[\frac{5}{8}\sqrt{s_F} - \frac{15 s_F}{8\pi\sqrt{2}} \sum_1^\infty\right.$$

$$\left.\frac{\sin(2\pi n s_F-\pi/4)}{n^{3/2}} - \frac{75}{32\pi^2\sqrt{2}} \sum_1^\infty \frac{\cos(2\pi n s_F-\pi/4)}{n^{5/2}}\right] .$$

The last term is safely neglected, since we have used $\epsilon_F(0)$ instead of $\epsilon_F(B)$ in the present argument, i.e., $s_F \gg 1$ in Eq. (2.21). Therefore we obtain

$$M_t = \frac{4N\mu_o^{5/2}m^{3/2}B^{3/2}}{3n\pi^2\hbar^3} \left[\sqrt{s_F} - \frac{3s_F}{\pi\sqrt{2}} \sum_1^{\infty} \frac{\sin(2\pi n s_F - \pi/4)}{n^{3/2}} \right].$$

(2.14")

This is the same as the result obtained by O'Connell[9] in his discussion of the magnetic white dwarf. His b is equal to our $2\pi s_F$.

Now differentiating Eq. (2.16) yields

$$\Lambda_{1/2}(s) = \frac{2}{3} s^{3/2} + \frac{1}{2} s^{1/2} + \frac{1}{24 s^{1/2}} + \frac{\sqrt{\pi}}{2} \sum_{P=2}^{\infty}$$

$$\frac{B_{2P}}{(2P)!\,\Gamma(5/2-2P)\,s^{2P-3/2}} + \frac{1}{2\sqrt{2\pi}} \sum_1^{\infty} \frac{\mathrm{sn}(2\pi n s - \pi/4)}{n^{3/2}} \tag{2.17}$$

Combining this Eq. (2.17) with Eq. (2.8) yields

$$N = \frac{B^{3/2}V}{\eta} \left[\frac{2}{3} s_F^{3/2} + \frac{\sqrt{\pi}}{2} \sum_1^{\infty} \frac{B_{2P}}{(2P)!\,\Gamma(5/2-2P)\,s_F^{2P-3/2}} \right.$$

$$\left. + \frac{1}{2\sqrt{2\pi}} \sum_1^{\infty} \frac{\sin(2\pi n s_F - \pi/4)}{n^{3/2}} \right]. \tag{2.18}$$

This was also obtained by O'Connell[9] with the second term neglected in the limit of $s_F >> 1$ (or $\varepsilon_F >> \hbar\omega_c$). A similar expression to our Eq. (2.17) also appears in the work by Gail et al.[10] Their expression now appears to be a poor approximation. Furthermore, their formulations use as the

Fermi energy $\varepsilon_F(0)$ for the study of the electron gas in magnetic fields. χ_t as a function of s_F has the following discontinuity at $s_F = k$:

$$\chi_t(k_+)-\chi_t(k_-) = \frac{9\mu_o N[1+2^{1/2}+\ldots+(k-1)^{1/2}+\frac{1}{2}k^{1/2}]^{5/3}}{(n\eta)^{2/3}[1+2^{1/2}+\ldots+(k-1)^{-1/2}+\frac{1}{2}k^{-1/2}]} \tag{2.19}$$

Fig. 7 shows pressure as a function of s_F. The three susceptibilities χ_t, χ_s, and χ_{orb} are shown in Fig. 8 in unit of χ_P, which confirms the discontinuities given by Eq. (2.19).

C. Ultra-Strong Field Limit

We first prove the following theorem.

<u>THEOREM</u>: The minimum field strength $B_\downarrow$, in which all the spins of the non-interacting electron gas are down, is given by $B_\downarrow = (2\eta n)^{2/3}$.

(Proof) If $B \geq B_\downarrow$ then $N_\downarrow = N$ and $M_s = \mu_o N$. However, $M_s = \frac{\mu_o N}{2\eta n} B^{3/2} s_F^{1/2}$; therefore, we obtain $s_F^{1/2} = 2\eta n\, B^{-3/2}$. Combining this with Eq. (2.3) we arrive at $\Lambda_{1/2}(s_F) = s_F^{1/2}$ which is possible if and only if $0 \leq s_F \leq 1$. Consequently we have proved that $B_\downarrow = B_o = (2\eta n)^{2/3}$.

For a metallic density[11] ($n \sim 10^{22}$), $B_{\downarrow} \sim 10^8$ Gauss, and for a semimetallic density[12] ($n \sim 10^{17}$), $B_{\downarrow} \sim 10^5$ Gauss. Now we focus on the values of B such that $B > B_o$ where s_F is less than unity. Then from Eq. (2.3) we know $s_F = 4\eta^2 n^2 B^{-3}$ and the subsequent expressions become extremely simple and can be obtained trivially as follows.

(i) $N = N_{\downarrow}$, $E_t = E_{\downarrow} = (\frac{8}{3})\mu_o \eta^2 n^2 N B^{-2}$, and $M_s = \mu_o N$

(ii) $P(B) = 16\mu_o \eta^2 n^3 / 3B^2$

(iii) $\chi_{s1} = 3\mu_o N/2B = -\chi_{s2}$, $\chi_s = 0$

(iv) $M_t = \frac{16}{3}\mu_o N \eta^2 n^2 B^{-3}$

(v) $\chi_t = -16\mu_o N \eta^2 n^2 B^{-4}$ (2.20)

The Fermi energy in this limit was obtained from Eq. (2.5) and shown to be proportional to B^{-2}. However, this can be obtained in another way using the above theorem. If $B \geq B_o$ then the single particle energy spectrum becomes one-dimensional and $E_0(k_z-1) = \hbar^2 k_z^2/2m$. Therefore it follows

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi ch}{eB}\right)^2 = \frac{2\pi^4 \hbar^4 c^2 n^2}{me^2}\frac{1}{B^2}$$

and

$$E_t = E_{\downarrow} = \left(\frac{eBL_1L_2}{hc}\right) \frac{L_3}{2\pi} \int_{-k_F}^{k_F} \frac{\hbar^2 k_z^2}{2m} \, dk_z = \frac{8}{3} \mu_o N \eta^2 n^2 B^{-2} ,$$

which is equal to $\frac{1}{3} \varepsilon_F N$ for the one-dimensional gas. This reduction of the three-dimensional electron gas to the one-dimensional gas is an important consequence of the ultra-strong field limit. All these results were exhibited in the preceeding figures in the interval $0<s_F \leq 1$.

D. Zero Field Limit

First we investigate the possibility of expanding the functions $E_t(B)$, $P(B)$, $M_t(B)$, and $\chi_t(B)$ as power series in B. Suppose we write

$$E_t(B) = E_t(0) + E_t'(0)\, B + \frac{E_t''(0)}{2!} B^2 + \ldots$$

$$= E_t(0) - M_t(0)B - \frac{1}{2} \chi_t(0)B^2 + \ldots .$$

Then we know that $M_t(0) = 0$ and $\chi_t(0)$ does not exist due to the factor of ds_F/dB in it (see Eq. (2.6)). Therefore, the power series does not exist for $E_t(B)$. The same arguments apply both to $M_t(B)$ and $P(B)$ also.

$$M_t(B) = M_t(0) + \chi_t(0)\, B + \ldots$$

and

$$P(B) = P(0) + P'(0)B + \dots ,$$

where $P'(0)$ is not defined because of the factor ds_F/dB. Therefore we should talk about the zero field limit only in the $T = 0^{\circ}K$ formalisms.

To obtain s_F as a function of B in the zero field limit we set

$$s_F (= \frac{\varepsilon_F(B)}{\hbar\omega_c}) \underset{B=0}{\sim} \frac{\varepsilon_F(0)}{\hbar\omega_c} = (\frac{3}{2}\eta n)^{2/3} \frac{1}{B} , \tag{2.21}$$

where

$$\varepsilon_F(0) = 2\mu_o(\frac{3}{2}\eta n)^{2/3} .$$

Fig. 9 shows the qualitative behavior of s_F as a function of B in both ultra-strong and weak field limits (an equivalent graph to Fig. 9 is found in Ref. 13).

The Eq. (2.21) now enables us to calculate $\chi_s(0)$ as $\chi_{s1} \underset{B=0}{\sim} \frac{3}{2}\chi_P$ where the Pauli spin susceptibility χ_P was given by $\chi_P = \frac{3}{4}\mu_o N(\frac{3}{2}\eta n)^{-2/3}$. Although $\chi_{s2}(0)$ and $\chi_t(0)$ are not well defined due to the factor ds_F/dB in it, we may obtain their average values on the B scale within a unit interval of s_F in the zero field limit as follows. First we write the asymptotic forms of the two functions $\Lambda_{1/2}(s)$ and $\Lambda_{3/2}(s)$ using the Euler-Maclaurin [EM] formula (see Chapter V for its definition);

$$\Lambda_{1/2}(s) \underset{\text{large } s}{\sim} \frac{2}{3} s^{3/2} + \frac{1}{2} s^{1/2} + \frac{1}{24} s^{-1/2} + 0(s^{-5/2})$$

(2.22)

$$\Lambda_{3/2}(s) \underset{\text{large } s}{\sim} \frac{2}{5} s^{5/2} + \frac{1}{2} s^{3/2} + \frac{1}{8} s^{1/2} + 0(s^{-3/2}) \ ,$$

(2.23)

which could have been written directly from Eqs. (2.16, 2.17). Now we define

$$\lim_{B\to 0} \langle \chi_{t(s2)} \rangle \equiv \lim_{k\to\infty} \frac{1}{B_k - B_{k-1}} \int_{B_{k-1}}^{B_k} \chi_{t(s2)} \, dB \ .$$

We then calculate

$$\lim_{B\to 0} \langle \chi_{s2} \rangle = \lim_{B\to 0} \frac{\mu_o N}{4\eta n(B_k - B_{k-1})} \int_k^{k+1} \frac{B^{3/2}}{s_F^{1/2}} \, ds_F$$

$$= \lim_{B\to 0} \frac{\mu_o N}{A(B_k - B_{k-1})} \int_k^{k+1} \frac{ds_F}{\sqrt{s_F}\,(\Lambda_{1/2} - \frac{1}{2}\sqrt{s_F})} \ .$$

Using Eqs. (2.21, 2.22), we arrive at $\lim_{B\to 0} \langle \chi_{s2} \rangle = -\frac{1}{2} \chi_P$.
Next we calculate $\langle \chi_t \rangle$ as follows.

$$\langle \chi_t \rangle = \frac{\mu_o N}{12(B_k - B_{k-1})} \int_k^{k+1} \frac{5(2\Lambda_{3/2} - s_F^{3/2})(1 - 2\sqrt{s_F}\Lambda_{-1/2}) + 36\sqrt{s_F}(\Lambda_{1/2} - \frac{1}{2}\sqrt{s_F})^2}{\sqrt{s_F}(\Lambda_{1/2} - \frac{1}{2}\sqrt{s_F})^2} ds_F$$

$$= \frac{\mu_o N}{12(B_k - B_{k-1})} \int_k^{k+1} \frac{-60ff'' + 36(f')^2}{(f')^2} ds = -60\{1 - [\frac{f}{F}]_k^{k+1}\} + 36,$$

where

$$f(s) = 2\Lambda_{3/2} - s^{3/2} \quad \text{and} \quad f'(s) = \frac{df}{ds} = 3(\Lambda_{1/2} - \frac{1}{2}\sqrt{s}).$$

Consequently we arrive at

$$\langle \chi_t \rangle = \frac{\mu_o N}{3(B_k - B_{k-1})} \{-6 + 5[\frac{2\Lambda_{3/2}(k) + (k+1)^{3/2}}{\Lambda_{1/2}(k) + \frac{1}{2}\sqrt{k+1}} - \frac{2\Lambda_{3/2}(k) - k^{3/2}}{\Lambda_{1/2}(k) - \frac{1}{2}\sqrt{k}}]\} .$$

This formula is exact. Now we employ Eqs. (2.21), (2.22), and (2.23) to obtain the zero field limit as

$$\lim_{B \to 0} \langle \chi_t \rangle = \frac{\mu_o N}{3(B_k - B_{k-1})} \{-6 + 5[\frac{3}{2}(\frac{4k}{5} + \frac{4}{5} + \frac{1}{5k} - \frac{3}{10k^2}) - \frac{3}{2}(\frac{4k}{5} + \frac{1}{5k})]\} = \frac{2}{3}\chi_P .$$

Collecting the results obtained so far, we now write

$$\chi_{s1} \underset{B=0}{\sim} \frac{3}{2}\chi_P, \quad \langle\chi_{s2}\rangle \underset{B=0}{\sim} -\frac{1}{2}\chi_P, \quad \langle\chi_s\rangle \underset{B=0}{\sim} \chi_P \quad ,$$

and

$$\langle\chi_t\rangle \underset{B=0}{\sim} \frac{2}{3}\chi_P ,$$

which in turn yields $\langle\chi_{orb}\rangle \underset{B=0}{\sim} -\frac{1}{3}\chi_P$ in this $T = 0^{\circ}K$ formalism.

However, it should be noticed that our system is somewhat non-physical since all the formulations started from the unattainable temperature $T = 0^{\circ}K$. To be more physical we may start our formulations at the non-zero temperatures and approach $T = 0^{\circ}K$ afterwards. In this latter case no discontinuities occur in any of the foregoing functions. Here we[6] recall the well-known formulas as below:

$$\chi_t' = \frac{4m^* \mu_o^2}{h^2}(3\pi^2 n)^{1/3}\left(1 - \frac{m^2}{3m^{*2}}\right)V \; ; \quad \chi_t' = \frac{2}{3}\chi_P, \; \chi_s' = \chi_P,$$

and

$$\chi_{orb}' = -\frac{1}{3}\chi_P$$

in case of $m^*=m$. Here we observe that

$$\chi_t' \equiv \lim_{B\to 0}\frac{M_t}{B} = \frac{dM_t}{dB}\Big]_{B=0} = \chi_t(0) \quad .$$

Unlike the case of the $T = 0^{\circ}K$ formalism, the above case does not require the averaging procedure, since all the functions are smooth due to the Fermi-Dirac factor. Finally we note the transition from

$$\left.\frac{\langle -\chi_{s2} \rangle}{\langle \chi_{s1} \rangle}\right]_{B=0} = \frac{1}{3} \quad \text{to} \quad \left.\frac{-\chi_{s2}}{\chi_{s1}}\right]_{B \geq B_{\downarrow}} = 1$$

as B increases.

CHAPTER III

THE DIRAC ELECTRON GAS

Although the exact solution of the Dirac electron in a uniform magnetic field has been known for nearly four[14-18] decades, the magnetic properties of a Dirac electron gas has not been throughly investigated. More recently Canuto et al.[19] studied the equation of state for this system at non-zero temperatures; however, their formulas are far from being exact due to the mathematical complications introduced by non-zero temperatures and their ignoring the B dependence of the Fermi energy $\varepsilon_F(B)$ entirely. The exact magnetic properties of the non-interacting Dirac electron gas can be derived at $T = 0^{\circ}K$ using the techniques developed in Chapter II for the Schrödinger gas.

Denoting the mechanical momentum by $\underset{\sim}{\pi}$, we write $\underset{\sim}{\pi} = \underset{\sim}{P} + e\underset{\sim}{A}/c$, where $\underset{\sim}{A} = (-\frac{yB}{2}, \frac{xB}{2}, 0)$ and $\underset{\sim}{B} = (0,0,B)$. This gives

$$\pi_{\pm} = \pi_x \pm i\pi_y = -i\hbar\left[\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y} + \frac{2}{\lambda^2}(x \pm iy)\right] ,$$

$$\pi_z = -i\hbar\frac{\partial}{\partial z} ,$$

where

$$\lambda^2 = \frac{\hbar c}{eB} .$$

Now the Hamiltonian reads

$$H = c\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi} + \beta mc^2$$

where

$$\underset{\sim}{\alpha} = ir_4\underset{\sim}{\gamma} = \begin{pmatrix} 0 , & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma} , & 0 \end{pmatrix} , \quad \beta = \gamma_4 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Now we notice that

$$[H,\sigma_3^o] = 0 ,$$

where

$$\sigma_3^o = \frac{\underset{\sim}{\sigma}^o\cdot B}{|\underset{\sim}{B}|} , \quad \underset{\sim}{\sigma}^o = \rho_3\underset{\sim}{\Sigma} + \frac{c\underset{\sim}{\pi}}{H+mc^2}\,\rho_1(1+\rho_3) ,$$ [20]

and

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & \underset{\sim}{\sigma} \end{pmatrix} .$$

Here

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \text{and} \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

This quantity σ_3^o is the spin component along the direction of $\underset{\sim}{B}$ field and is the only component which commutes with the Hamiltonian H; i.e., it is a constant of motion. Therefore we construct the simultaneous eigenfunctions of H and σ_3^o , where σ_3^o is now given by

$$\sigma_3^o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{2c\pi_3}{H+mc^2} & 0 & -1 & 0 \\ 0 & \frac{2c\pi_3}{H+mc^2} & 0 & +1 \end{pmatrix}$$

$HU_{n\sigma} = E_{n\sigma}U_{n\sigma}$ and $\sigma_3^o U_{n\sigma} = \sigma U_{n\sigma}$. Here $\sigma = \pm 1$ are the spin components along the direction of the $\underset{\sim}{B}$ field. The eigenfunctions are given by

$$U_{n\uparrow} = G_\uparrow \begin{bmatrix} (E_{n\uparrow}+mc^2)\phi_n \\ 0 \\ c\hbar\, k_z\ \phi_n \\ \varepsilon(n+1)^{1/2}\ \phi_{n+1} \end{bmatrix}$$

and

$$U_{n\downarrow} = G_\downarrow \begin{bmatrix} 0 \\ (E_{n\downarrow}+mc^2)\phi_n \\ \varepsilon n^{1/2}\ \phi_{n-1} \\ -c\hbar\, k_z\ \phi_n \end{bmatrix},$$

where $G_\sigma = [2E_{n\sigma}(E_{n\sigma}+mc^2)]^{-1/2}$, $\varepsilon^2 = 2ce\hbar B$, and $k_z = \frac{2\pi}{L} n_z$ $(n_z = 0, \pm 1, \pm 2 \ldots)$. The energy eigenvalues are given by

$$E_{n\sigma}(k_2) = (m^2c^4+c^2\hbar^2k_z^2+\varepsilon^2(n+\tfrac{1}{2})+\tfrac{1}{2}\ \varepsilon^2\sigma)^{1/2}\ .$$

More specifically,

$$E_{n\uparrow}(k_z) = (m^2c^4 + c^2\hbar^2k_z^2 + \varepsilon^2(n+1)]^{1/2} ,$$

$$E_{n\downarrow}(k_z) = [m^2c^4 + c^2\hbar^2k_z^2 + \varepsilon^2 n]^{1/2} , \quad n = 0,1,2,\ldots.$$

The function $\phi_n(\underset{\sim}{r})$ can be written in two different coordinate frames, Cartesian and cylindrical, as below.

(i) Cartesian coordinates:

$$\phi_{n,a}(x,y,z) = N_n H_n(\frac{x-a}{\lambda}) \exp(ik_z z) \exp(iy \frac{x-2a}{2\lambda^2})$$

$$\exp[- \frac{(x-a)^2}{2\lambda^2}] ,$$

where $x_o\phi_{n,a} = a\phi_{n,a}$, $a = p_y c/eB$, and (x_o,y_o) is the coordinate of the orbit center.

(ii) Cylindrical coordinates:

$$\phi_{n,\ell}(r,\phi,z) = N_n L_n^{\ell}(\rho)\rho^{(n-\ell)/2} \exp(- \frac{\rho}{2})$$

$$\exp(ik_z z) \exp[i(n-\ell)\phi] ,$$

where

$$N_n = (-i)^n [\tfrac{1}{2}\pi\ L_3 \ell!n!]^{-1/2}, \ \rho = \gamma^2/2\lambda^2, \ \ell = 0,1,2,\ldots,$$

$$\gamma_o^2\phi_{n,\ell} = \lambda\sqrt{2\ell+1}\ \phi_{n,\ell}, \text{ and } \gamma_o^2 = x_o^2 + y_o^2 .$$

The solution (i) is degenerate with respect to a uniform distribution of circular orbits with the same energy whose center lies on the line x=a. This corresponds to the classical limit that the axes of the helical orbits of the electron may lie anywhere on the line x=a. The wave-function (ii) is degenerate with respect to a distribution of orbits whose centers lie on the circle of radius $\gamma_o = \lambda\sqrt{2\ell+1}$.

Now we subtract the rest mass energy mc^2 from $E_{n\sigma}(k_z)$ and write

$$E_{n\sigma}(k_z) = (m^2c^4 + c^2\hbar^2k_z^2 + \varepsilon^2(n+\tfrac{1}{2}) + \tfrac{1}{2}\,\sigma\varepsilon^2)^{1/2} - mc^2 \tag{3.1}$$

with the degeneracy

$$g_E = \frac{2eBL_1L_2}{hc} \; .$$

Defining a new energy spectrum $T_{n\sigma}(k_z)$ by

$$T_{n\sigma}(k_z) \equiv \frac{E_{n\sigma}}{2mc^2}\,(E_{n\sigma}+2mc^2) \tag{3.2}$$

we immediately obtain

$$T_{n\sigma}(k_z) = \frac{\hbar^2k_z^2}{2m} + (n+\tfrac{1}{2})\hbar\omega_c + \sigma\mu_o B \; , \tag{3.3}$$

which is just equal to $E_n(k_z,\sigma)$ in the Schrödinger case. Eq. (3.2) can be rewritten as

$$E_{n\sigma}(k_z) = mc^2\left(\sqrt{1 + \frac{2T_{n\sigma}(k_z)}{mc^2}} - 1\right) . \tag{3.4}$$

First we notice that

$$E_{n\sigma}(k_z) \underset{\text{NR limit}}{\frown} T_{n\sigma}(k_z) \qquad \text{from Eq. (3.4).}$$

Since the functions $E_{n\sigma} = E_{n\sigma}(T_{n\sigma})$ or $T_{n\sigma} = T_{n\sigma}(E_{n\sigma})$ are single-valued, we see that $N(E) = N(T)$ and $g_E = g_T$, where $N(E)$ [or $N(T)$] is the total number of quantum states whose eigenvalues $E_{n\sigma}$ (or $T_{n\sigma}$) are below or equal to E (or T) and g_E (or g_T) is the degeneracy in the E (or T) spectrum. As in the case of the Schrödinger theory we easily obtain

$$N(T) = \frac{2eBV(2m)^{1/2}}{ch^2} \sum_{0}^{[\ ]} [T-(n+\tfrac{1}{2})\hbar\omega_c]^{1/2} ,$$

where

$$[\ ] = [\frac{T}{\hbar\omega_c} - \frac{1}{2}] ,$$

$$D(T) = \frac{dN(T)}{dT} = A \sum_{0}^{[\ ]} (s-n)^{-1/2} ,$$

where

$$s = \frac{T}{\hbar\omega_c} - \frac{1}{2} \quad \text{and} \quad A = \frac{eBV}{\hbar^2 c}\left(\frac{2m}{\hbar\omega_c}\right)^{1/2} .$$

Consequently we write

$$N(E) = N(T) = 2\hbar\omega_c \, A \, \Lambda_{1/2}(s) \; ,$$

$$D(T) = A \, \Lambda_{-1/2}(s) .$$

Finally the density of states D(E) is given by

$$D(E) = \frac{dN(E)}{dE} = \frac{dN(T)}{dT}\frac{dT}{dE} = D(T) \quad \sqrt{\frac{2}{mc^2}\left(T + \frac{mc^2}{2}\right)} \; . \tag{3.5}$$

We here stress that all the subsequent formulations are stated always in the rest frame of the thermodynamic system itself.

In Appendix B, we derive Onsager's flux quantization using the present Dirac theory.

A. Exact Relativistic Fermi Energy $\varepsilon_F(B)$

Since σ_3^o is a constant of motion, $N_\uparrow$ and $N_\downarrow$ will also be constants of motion. These are given by

$$N_{\uparrow} = \int_{\mu_o B + \hbar\omega_c/2}^{t_F} D(T-\mu_o B)\,dT = Ah/\omega_c \int_0^{s_F - 1} \Lambda_{-1/2}(s)\,ds \ ,$$

$$N_{\downarrow} = \int_{-\mu_o B + \hbar\ _c/2}^{t_F} D(T+\mu_o B)\,dT = A\hbar\omega_c \int_0^{s_F} \Lambda_{-1/2}(s)\,ds \ ,$$

where

$$s_F = \frac{t_F}{\hbar\omega_c} \qquad \text{and} \qquad t_F = \frac{\varepsilon_F}{2mc^2}\,(\varepsilon_F + 2mc^2) \ .$$

Consequently from $N = N_{\uparrow} + N_{\downarrow}$ we arrive at

$$\Lambda_{1/2}(s_F) = \frac{1}{2}\, s_F^{1/2} + \eta n B^{-3/2} \ , \tag{3.6}$$

which is the same as Eq. (2.3) of the Schrödinger case. Fig. 10 shows s_F as a function of B for $n = 10^{28}$ and 10^{34} cm^{-3}. However, one should notice that

$$\varepsilon_F = mc^2 \left(\sqrt{1 + \frac{2t_F}{mc^2}} - 1\right) = t_F - \frac{t_F^2}{2mc^2} + \ldots \underset{\text{NR limit}}{\frown} t_F \tag{3.7}$$

or equivalently

$$\varepsilon_F = mc^2 \left(\sqrt{1 + a s_F} - 1\right) \underset{\text{NR limit}}{\frown} \hbar\omega_c\, s_F \ , \tag{3.7'}$$

where $a = 2\hbar\omega_c/mc^2$. Notice that $t_F \equiv \varepsilon_F(B)]$ Schrödinger. Therefore Eq. (3.6) determines s_F for given values of B and n, and s_F in turn gives $\varepsilon_F(B)$ for those B and n through Eq. (3.7'). Fig. 11 compares $\varepsilon_F(B)_D$, $\varepsilon_F(B)_s$, $\varepsilon_F(0)_D$, and $\varepsilon_F(0)_s$ for the electronic density $n = 10^{34}\ \text{cm}^{-3}$, where D and s denote Dirac and Schrödinger, respectively. Here $\varepsilon_F(0)_s = 2\mu_o(\frac{3}{2}\eta n)^{2/3}$ and $\varepsilon_F(0)_D = mc^2[(1+x_F^2)^{1/2}-1]$ by Eq. (D.3) in Appendix D. As is shown there for the higher density, the Dirac theory deviates considerably from the Schrödinger theory since relativistic effects become significant. We also obtain the same formulas as Eqs. (2.4) and (2.7). Next we observe that

(i) if $0 < s_F \leq 1$, then $\Lambda_{1/2}(s_F) = s_F^{1/2}$ and Eq. (3.6) gives

$$t_F(B) = 8\eta^2 n^2 \mu_o B^{-2} \quad \text{and} \quad \varepsilon_F(B) = mc^2[(1 + \frac{16\eta^2 n^2 \mu_o}{mc^2\ B^2})^{1/2} -1] \ . \tag{3.8}$$

(ii) if $1 < s_F \leq 2$, then $\Lambda_{1/2}(s_F) = s_F^{1/2}+(s_F-1)^{1/2}$ and we derive the same equation determining s_F in this interval as Eq. (2.5). In principle, this method can be extended to yield the equations determining s_F of higher values.

B. Susceptibilities

In Dirac theory, the transverse spin components also exist, but these are not constants of motion. As was discussed earlier, only the longitudinal spin component is a constant of motion. This naturally defines the spin magnetization M_s as

$$M_s = \mu_o(N_\downarrow - N_\uparrow) = \frac{\mu_o N}{2\eta n} B^{3/2} s_F^{1/2} .$$

The spin susceptibility χ_s follows easily.

$$\chi_s = \frac{\partial M_s}{\partial B} = \frac{3\mu_o N}{4\eta n} (Bs_F)^{1/2} + \frac{\mu_o N}{4\eta n} \frac{B^{3/2}}{s_F^{1/2}} \frac{ds_F}{dB}$$

or

$$\chi_s = \frac{3\mu_o N}{4\eta n} (Bs_F)^{1/2} - \frac{3}{2} \frac{\mu_o N}{B(2s_F^{1/2}\Lambda_{-1/2}(s_F)-1} = \chi_{s1} + \chi_{s2}. \tag{3.9}$$

These are exactly the same as those in the Schrödinger case. Consequently we should expect the same discontinuities in $\chi_s(s_F)$ as in Formula (2.10). Next we derive E_t, M_t, and χ_t as follows. $E_t = E_\uparrow + E_\downarrow$, where

$$E_\uparrow = mc^2 \int_{\mu_o B + \hbar\omega_c/2}^{t_F} \left(\sqrt{1 + \frac{2T}{mc^2}} - 1\right) D(T-\mu_o B)\, dT$$

$$= A\hbar\omega_c mc^2 \int_0^{s_F - 1} [\sqrt{1+a(s+1)} - 1]\, \Lambda_{-1/2}(s)\, ds$$

and

$$E_{\downarrow} = mc^2 \int_{-\mu_o B + \hbar\omega_c/2}^{t_F} (\sqrt{1+2T/mc^2} - 1) D(T+\mu_o B)\, dT$$

$$= A\hbar\omega_c\, mc^2 \int_0^{s_F} (\sqrt{1+as} - 1)\Lambda_{-1/2}(s)\, ds \ .$$

Now we define a new function $\Sigma_{-1/2}(a;s)$ by

$$\Sigma_{-1/2}(a;s) \equiv \int_0^s \sqrt{1+at}\, \Lambda_{-1/2}(t)\, dt \ .$$

Using the definition of the function $\Lambda_{-1/2}(s)$, $\Sigma_{-1/2}(a;s)$ is expanded as

$$\Sigma_{-1/2}(a;s) \equiv \int_0^s \sqrt{1+at}\, \Lambda_{-1/2}(t)\, dt = \int_0^1 \sqrt{1+at}\, \frac{1}{\sqrt{t}}\, dt$$

$$+ \int_1^2 \sqrt{1+at}\, \left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t-1}}\right) dt$$

$$+ \ldots + \int_{[s]}^s \sqrt{1+at}\, \left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t-1}} + \ldots + \frac{1}{t-[s]}\right) dt$$

$$= \int_0^s \frac{1+at}{t}\, dt + \int_0^{s-1} \sqrt{\frac{1+a(t+1)}{t}}\, dt$$

$$+ \ldots + \int_0^{s-[s]} \sqrt{\frac{1+a(t+[s])}{t}}\, dt \ .$$

By a change of variable $x = \sqrt{t}$, these integrals are re-written as

$$\Sigma_{-1/2}(a;s) = 2\{\int_0^{\sqrt{s}} \sqrt{1+ax^2}\,dx + \int_0^{\sqrt{s-1}} \sqrt{1+a(x^2+1)}\,dx$$

$$+ \dots + \int_0^{\sqrt{s-[s]}} \sqrt{1+a(x^2+[s])}\,dx\ \}$$

$$= 2 \sum_0^{[s]} \int_0^{\sqrt{s-n}} \sqrt{1+an+ax^2}\,dx\ .$$

This is easily integrated and finally leads to

$$\Sigma_{-1/2}(a;s) = \sum_0^{[s]} \{\sqrt{(s-n)(1+as)} + \frac{1+an}{\sqrt{a}}$$

$$[\ell n(\sqrt{a(s-n)} + \sqrt{1+as}) - \ell n\sqrt{1+an}\ ]\ \}. \tag{3.10}$$

Furthermore, we find from Eq. (3.10) the expressions for $(\partial\Sigma_{-1/2}/\partial a)_s$ and $(\partial^2\Sigma_{-1/2}/\partial a^2)_s$ as

$$(\frac{\partial\Sigma_{-1/2}}{\partial a})_s = \sum_0^{[s]} \{\ \frac{1}{2a}\sqrt{(s-n)(1+as)} + \frac{an-1}{2a^{3/2}}$$

$$[\ell n(\sqrt{a(s-n)} + \sqrt{1+as}) - \ell n\sqrt{1+an}\ ]\quad , \tag{3.11}$$

$$\left(\frac{\partial^2\Sigma_{-1/2}}{\partial a^2}\right)_s = \sum_{0}^{[s]} \left\{- \frac{(1+as+3an+a^2ns)\sqrt{s-n}}{4a^2(an+1)\sqrt{1+as}} + \frac{3-an}{4a^{5/2}}\right.$$

$$\left.[\ell n(\sqrt{a(s-n)} + \sqrt{1+as}) - \ell n\sqrt{1+an}\,] - \frac{n(an-1)}{2a^{3/2}(an+1)}\right\} .$$

We further notice that the functions $\Sigma_{-1/2}$ and $(\partial\Sigma_{-1/2}/\partial a)_s$ are continuous; however, the function $\Sigma''_{-1/2}(s) \equiv (\partial^2\Sigma_{-1/2}/\partial a^2)_s$ has the discontinuity at s=k

$$\Sigma''_{-1/2}(k_+) - \Sigma''_{-1/2}(k_-) = - \frac{1+4ak}{4a^2(ak+1)^{3/2}} - \frac{k(ak-1)}{2a^{3/2}(ak+1)} .$$

Next we obtain the following from the definition of $\Sigma_{-1/2}(a;s)$.

$$\left(\frac{\partial\Sigma_{-1/2}}{\partial s}\right)_a = \sqrt{1+as}\ \Lambda_{-1/2}(s) \tag{3.13}$$

$$\Sigma_{-1/2}(0;s) = 2\Lambda_{1/2}(s) \ , \quad \Sigma_{-1/2}(a;0) = 0$$

$$\Sigma_{-1/2}(a;s) = \sqrt{s(1+as)} + \frac{1}{\sqrt{a}} \sinh^{-1}\sqrt{as} \ , \ \text{if } 0<s\leq 1$$

(3.14)

$$\int_0^{s-1} \sqrt{1+a(t+1)}\ \Lambda_{-1/2}(t)dt = \Sigma_{-1/2}(a;s) - \sqrt{s(1+as)}$$

$$- \frac{1}{\sqrt{a}} \sinh^{-1}\sqrt{as}$$

Fig. 12 shows the functions $\Sigma_{-1/2}(a;s)$ for a = 1.5, and 10. Using the above formulas we obtain

$$E_{\uparrow} = A\hbar\omega_c mc^2 \; [\Sigma_{-1/2}(a;s_F) - \sqrt{s_F(1+as_F)}$$

$$- \frac{1}{a} \sinh^{-1} \sqrt{as_F} - 2\Lambda_{1/2}(s_F) + 2\sqrt{s_F} \;] \; ,$$

$$E_{\downarrow} = A\hbar\omega_c mc^2 \; [\Sigma_{-1/2}(a;s_F) - 2\Lambda_{1/2}(s_F)] \; .$$

Therefore we arrive at

$$E_t = \frac{\mu_o N}{\eta n} B^{5/2} \; f(s_F) \; , \tag{3.15}$$

where

$$f(s_F) = \frac{2}{a} \Sigma_{-1/2}(a;s_F) - \frac{4}{a} \eta n B^{-3/2}$$

$$- \frac{\sqrt{s_F(1+as_F)}}{a} - a^{-3/2} \sinh^{-1} \sqrt{as_F} \; .$$

The pressure is then given by

$$P(B) = - \frac{\partial E_t}{\partial V} = \frac{4\mu_o nB\sqrt{1+as_F}}{a} - \frac{\mu_o B^{5/2}}{\eta} \; [\frac{2}{a} \Sigma_{-1/2}(a;s_F)$$

$$- \frac{1}{a} \sqrt{s_F(1+as_F)} - a^{-3/2} \sinh^{-1} \sqrt{as_F} \;] \; , \tag{3.16}$$

where the formula (2.13) has been used. The pressures $P_D(B)$, $P_s(B)$, $P_D(0)$, and $P_s(0)$ are shown in Fig. 13 as functions of s_F for the electronic density $n = 10^{28}$ cm^{-3}. Here $P_s(0) = \frac{4}{5}\mu_o n(\frac{3}{2}\eta n)^{2/3}$ and $P_D(0)$ is given by Eq. (D.2) in Appendix D.

Now the total magnetization and susceptibility are given by

$$M_t = -\frac{\partial E_t}{\partial B} = -\frac{\mu_o N}{\eta n} B^{3/2}\Big(\frac{3}{a}\Sigma_{-1/2} - \frac{2}{a}\sqrt{s_F(1+as_F)}$$

$$+ 2\Big(\frac{\partial\Sigma_{-1/2}}{\partial a}\Big)_{s_F} - \frac{6\eta n\sqrt{1+as_F}}{aB^{3/2}} \tag{3.17}$$

$$- a^{-3/2}\sinh^{-1}\sqrt{as_F}\Big),$$

$$\chi_t = \frac{\partial M_t}{\partial B} = \frac{3\mu_o N}{\sqrt{1+as_F}}\Big(\frac{ds_F}{dB}\Big) + \frac{3\mu_o N(1+3as_F)}{aB\sqrt{1+as_F}} - \frac{\mu_o N}{\eta n}B^{1/2}$$

$$\Big[\frac{3\Sigma_{-1/2}}{2a} + 6\Big(\frac{\partial\Sigma_{-1/2}}{\partial a}\Big)_{s_F} + 2a\Big(\frac{\partial^2\Sigma_{-1/2}}{\partial a^2}\Big)_{s_F}$$

$$- \frac{\sqrt{s_F}(3+4as_F)}{2a\sqrt{1+as_F}}\Big], \tag{3.18}$$

where we used Eqs. (3.6), (3.13), and the formula

$$\frac{df(s_F)}{dB} = \frac{1}{2aB}\big(\sqrt{s_F(1+as_F)} - 4\Sigma_{-1/2}\big) + \frac{2}{B}\Big(\frac{\partial\Sigma_{-1/2}}{\partial a}\Big)_{s_F}$$

$$+ \frac{3}{2B}a^{-3/2}\sinh^{-1}\sqrt{as_F} + \frac{2\eta nB^{-5/2}}{a}\big(5-3\sqrt{1+as_F}\big).$$

The discontinuity of $\chi_t(s_F)$ at $s_F = k$ is found to be

$$\chi_t(k_+)-\chi_t(k_-) = \frac{9\mu_o N[1+\sqrt{2}+\ldots+\sqrt{k-1}+\frac{1}{2}\sqrt{k}]^{5/3}}{(\eta n)^{2/3}\sqrt{1+ak}\,[1+2^{-1/2}+\ldots+(k-1)^{-1/2}+\frac{1}{2}k^{-1/2}]}$$

$$+\frac{\mu_o N}{(\eta n)^{2/3}}\,\frac{1+4ak+2\sqrt{a}\,k(ak-1)(1+ak)^{1/2}}{2a(1+ak)^{3/2}[1+\sqrt{2}+\ldots+\sqrt{k-1}+\frac{1}{2}\sqrt{k}]^{1/3}}\ . \tag{3.19}$$

The functions $(\partial\Sigma_{-1/2}/\partial a)_s$ and $(\partial^2\Sigma_{-1/2}/\partial a^2)_s$ are presented in Fig. 14, 15. However, one should notice that these functions depend solely on B since $a = \frac{4\mu_o}{mc^2}B$ and $s_F = s_F(B)$ for a given density n. The functions $\Sigma_{-1/2}(a;s_F)$ and $(\partial\Sigma_{-1/2}/\partial a)_{s_F}$ as functions of s_F for $n = 10^{28}$ cm^{-3}, are shown in Fig. 16.

C. Ultra-Strong Field Limit

We first state the following theorem.

<u>THEOREM</u> The minimum field strength $B_\downarrow$ in which all the spins (in their rest frames) of the non-interacting Dirac electron gas are down is given by $B_\downarrow = (2\eta n)^{2/3}$.

The proof is the same as in the Schrödinger case.

Now we focus on the values of B such that $B \geq B_o$, where $s_F \leq 1$, and then Eq. (3.6) gives $s_F = 4\eta^2n^2B^{-3}$. The formulas become extremely simple and can be obtained trivially as

$$\text{(i)} \quad N = N_{\downarrow} \;, \; E_t = E_{\downarrow} = \frac{\mu_o N}{\eta n} B^{5/2} f_{\downarrow}(x), \quad \text{and} \quad M_s = \mu_o N,$$

where

$$f_{\downarrow}(x) = \frac{1}{a} [\sqrt{x(1+ax)} - 2\sqrt{x} + \frac{1}{a^{1/2}} \sinh^{-1} \sqrt{ax}] \;,$$

$$x \equiv s_F(\leq 1) = 4\eta^2 n^2 B^{-3} \;,$$

$$\text{(ii)} \quad P(B) = \frac{4\mu_o nB}{a} \sqrt{1+ax} - \frac{\mu_o B^{5/2}}{\eta a^{3/2}} [\sqrt{ax(1+ax)} + \sinh^{-1} \sqrt{ax}] \;,$$

$$\text{(iii)} \quad \chi_{s1} = 3\mu_o N/2B = -\chi_{s2} \quad , \quad \chi_s = 0 \;,$$

$$\text{(iv)} \quad M_t = \frac{\mu_o N}{a\eta n} B^{3/2} \left(\sqrt{x(1+ax)} - \frac{1}{a^{1/2}} \sinh^{-1} \sqrt{ax}\right) \;,$$

$$\text{(v)} \quad \chi_t = - \frac{2\mu_o N x^2 B^{1/2}}{\eta n \sqrt{x(1+ax)}} = - \frac{16\eta^2 n^2 \mu_o N B^{-4}}{\sqrt{1+4a\eta^2 n^2 B^{-3}}} \;,$$

where we used formula (3.14) in obtaining $f_{\downarrow}(x)$. The above results are also obtainable in another way. If $B \geq B_{\downarrow}$, then the T-spectrum becomes one-dimensional. $T_o(k_z,-1) = \frac{\hbar^2 k_z^2}{2m}$ and therefore we get

$$t_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi ch}{eB}\right)^2 = 8\eta^2 n^2 \mu_o B^{-2},$$

which gives

$$\varepsilon_F(B) = mc^2\left(\sqrt{1 + \frac{16\eta^2 n^2 \mu_o}{B^2 mc^2}} - 1\right)$$

and

$$s_F(B) = 4\eta^2 n^2 B^{-3} .$$

Finally

$$E_t = E_\downarrow = \frac{eBL_1L_2}{hc}\left(\frac{L_3}{2\pi}\right)\int_{-k_F}^{k_F} mc^2\left(\sqrt{1 + \left(\frac{\hbar k_z}{mc}\right)^2} - 1\right)dk_z$$

$$= \frac{\mu_o N}{\eta n} B^{5/2} f_\downarrow(x) ,$$

where $x = 4\eta^2 n^2 B^{-3}$.

D. Zero Field Limit

Following the same arguments as in Section D of Chapter II we consider only the zero field limit. Setting

$$\varepsilon_F(B) \underset{B=0}{\sim} mc^2(\sqrt{1+x_F^2} - 1),$$

where

$$x_F = \frac{\hbar(3\pi^2 n)^{1/3}}{mc} \,.$$

This is given by Eq. (D.3) in Appendix D. This in turn yields

$$s_F \underset{B=0}{\sim} \frac{X_F^2}{a} = \left(\frac{3}{2}\,\eta n\right)^{2/3} \frac{1}{B} \,, \tag{3.21}$$

which is exactly the same as Eq. (2.21) in the case of the Schrödinger gas. The asymptotic expansion of the function $\Sigma_{-1/2}(a;s)$ in Eq. (3.10) is obtained by the use of the EM formula as

$$\Sigma_{-1/2}(a;s) \underset{\text{large } s}{\sim} \sqrt{s(1+as)}\left(\frac{1}{2a} + s + \frac{1}{2}\right) + \frac{a-1}{2a^{3/2}} \sinh^{-1}\sqrt{as} \,. \tag{3.22}$$

This Eq. (3.23) readily generates

$$f(s_F) \underset{B=0}{\sim} a^{-5/2}\left[x_F(1+2x_F^2)(1+x_F^2)^{1/2} - \sinh^{-1} x_F - \frac{8}{3}x_F^3\right] , \tag{3.23}$$

$$\left(\frac{\partial\Sigma_{-1/2}}{\partial a}\right)_s \underset{\text{large } s}{\sim} \frac{\sqrt{s}}{4a^2\sqrt{1+as}}\left[2(as)^2 + (a-1)as + a-3\right] + \frac{3-a}{4a^{5/2}} \sinh^{-1}\sqrt{as} \,, \tag{3.24}$$

and

$$\left(\frac{\partial^2 \Sigma_{-1/2}}{\partial a^2}\right)_s \underset{\text{large } s}{\sim} \frac{-\sqrt{s}}{8a^3(1+as)^{3/2}} [2(as)^3 + (a-3)(as)^2$$

$$+ 4(a-5)as + 3a-15] + \frac{3(a-5)}{8a^{7/2}} \sinh^{-1} \sqrt{as} \; .$$

(3.25)

Using all of these formulas we finally arrive at

$$E_t(B) \underset{B=0}{\sim} \frac{V(mc^2)^{5/2}}{32\eta\mu_o^{3/2}} [x_F(1+2x_F^2)(1+x_F^2)^{1/2} - \sinh^{-1} s_F$$

$$- \frac{8}{3} x_F^3] \; , \tag{3.26}$$

$$P(B) \underset{B=0}{\sim} \frac{(mc^2)^{5/2}}{96\eta\mu_o^{3/2}} [x_F(2x_F^2-3)(1+x_F^2)^{1/2} + 3 \sinh^{-1} x_F] \; ,$$

(3.27)

$$M_t(B) \underset{B=0}{\sim} 0 \; . \tag{3.28}$$

The above Eqs. (3.26) and (3.27) are the same as Eqs. (D.1) and (D.2) in Appendix D as should be the case.

E. Non-Relativistic Limit

We first notice that the limit $\frac{\varepsilon_F}{mc^2} \to 0$ is equivalent to $as_F \to 0$. Therefore

$$\underset{\text{NR limit}}{\sim} \; = \; \underset{as_F \to 0}{\sim} \; .$$

Eq. (3.13) gives

$$\left(\frac{\partial \Sigma_{-1/2}}{\partial s_F}\right)_a \underset{\text{NR limit}}{\sim} \Lambda_{-1/2}(s_F) + \frac{1}{2}\, as_F\, \Lambda_{-1/2}(s_F) .$$

This yields

$$\Sigma_{-1/2}(a;s_F) \underset{\text{NR limit}}{\sim} 2\Lambda_{1/2}(s_F) + as_F\Lambda_{1/2}(s_F) - \frac{2}{3}\, a\Lambda_{3/2}(s_F) \; . \tag{3.29}$$

By means of formula (3.29) we write

$$f(s_F) \underset{\text{NR limit}}{\sim} 2s_F\Lambda_{1/2}(s_F) - \frac{4}{3}\,\Lambda_{3/2}(s_F) - \frac{1}{3}\, s_F^{3/2} \; . \tag{3.30}$$

This enables us to write

$$E_t \underset{\text{NR limit}}{\sim} \frac{\mu_o N}{\eta n} B^{5/2} \left[2s_F\Lambda_{1/2}(s_F) - \frac{4}{3}\,\Lambda_{3/2}(s_F) - \frac{1}{3}\, s_F^{3/2}\right]$$

which is already established by Eq. (2.11). The pressure P(B) in this limit is given by

$$P(B) = \frac{2\mu_o}{3\eta} B^{5/2} \, [2\Lambda_{3/2}(s_F) - s_F^{3/2}] \ ,$$

which is the same as Eq. (2.12). From Eq. (3.29) we derive

$$\left(\frac{\partial \Sigma_{-1/2}}{\partial a}\right)_{s_F} \underset{\text{NR limit}}{\frown} s_F\Lambda_{1/2}(s_F) - \frac{2}{3} \Lambda_{3/2}(s_F) \ , \tag{3.31}$$

$$\left(\frac{\partial^2 \Sigma_{-1/2}}{\partial a^2}\right)_{s_F} \underset{\text{NR limit}}{\frown} 0 \ . \tag{3.32}$$

Finally we arrive at

$$M_t \underset{\text{NR limit}}{\frown} \frac{5}{3} \frac{\mu_o N}{\eta n} B^{3/2} \{2\Lambda_{3/2} - s_F^{3/2}\} - 2\mu_o N s_F$$

and

$$\chi_t \underset{\text{NR limit}}{\frown} 3\mu_o N \frac{ds_F}{dB} + \frac{5}{2} \frac{\mu_o N}{\eta n} B^{1/2} \{2\Lambda_{3/2} - s_F^{3/2}\} \ ,$$

which are just Eqs. (2.14) and (2.15) as should be the case.

Next we obtain the ultra-strong and NR limit. By the use of the formula

$$\sinh^{-1}x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$$

the quantities in Eq. (3.20) are expanded in the limit of $as_F \to 0$ as

$$E_t = \frac{\mu_o N}{a\eta n} B^{5/2} \left[\sqrt{x}\left(1+\frac{1}{2}ax-\frac{1}{8}a^2x^2+\ldots\right) - 2\sqrt{x} + \frac{1}{\sqrt{a}}\right.$$

$$\left.\left(\sqrt{ax}-\frac{1}{6}ax\sqrt{ax}+\frac{3\sqrt{ax}}{40}a^2x^2-\ldots\right)\right]$$

$$= \frac{8\eta^2n^2\mu_oN}{3B^2} - \frac{32\mu_o^2N\eta^4n^4}{5mc^2B^4} + \ldots ,$$

$$M_t = \frac{\mu_o N}{a\eta n} B^{3/2} \left[\sqrt{x}\left(1+\frac{1}{2}ax-\frac{1}{8}a^2x^2+\ldots\right)\right.$$

$$\left. - \frac{1}{\sqrt{a}}\left(\sqrt{ax}-\frac{1}{6}ax\sqrt{ax}+\frac{3\sqrt{ax}}{40}a^2x^2-\ldots\right)\right]$$

$$= \frac{16\eta^2n^2\mu_oN}{3B^3} - \frac{128\eta^4n^4\mu_o^2N}{5mc^2B^5} + \ldots ,$$

$$P(B) = \frac{2\mu_o nB}{a}\left(1+\frac{1}{2}ax-\frac{1}{8}a^2x^2+\ldots\right) - \frac{\mu_o B^{5/2}}{\eta a^{3/2}}$$

$$\left[\sqrt{ax}-\frac{1}{6}ax\sqrt{ax}+\frac{3\sqrt{ax}}{40}a^2x^2-\ldots\right]$$

$$= \frac{16\mu_o\eta^2n^3}{3B^2} - \frac{128\mu_o^2\eta^4n^5}{5B^4mc^2} + \ldots ,$$

$$\chi_t = \frac{-2\mu_o N x^2 B^{1/2}}{\eta n\sqrt{x}} \left(1-\frac{1}{2}ax+\ldots\right) = -\frac{16\eta^2 n^2 \mu_o N}{B^4}$$

$$+\frac{128\mu_o^2 N\eta^4 n^4}{mc^2 B^6} + \ldots ,$$

where the first terms in the above expansions were already established in the Schrödinger case, Eq. (2.20).

Finally, we also obtain the zero-field and NR limit for the total energy E_t and the pressure P_t. Equations (3.26) and (3.27) are expanded in the limit of $\sqrt{as_F} \equiv x_F << 1$ as

$$E_t(0) = \frac{V(mc^2)^{5/2}}{32\eta\mu_o^{3/2}} \left[x_F(1+2x_F^2)\left(1+\frac{1}{2}x_F^2-\frac{1}{8}x_F^4+\frac{1}{16}x_F^6+\ldots\right)\right.$$

$$\left.-\left(x_F-\frac{1}{6}x_F^3+\frac{3}{40}x_F^5-\frac{5}{112}x_F^7\ldots\right) - \frac{8}{3}x_F^3\right]$$

$$= \frac{V\hbar^5(3\pi^2 n)^{5/3}}{40\eta\mu_o^{3/2}m^{5/2}} - \frac{Vm^{5/2}\hbar^7(3\pi^2 n)^{7/3}}{224\eta\mu_o^{3/2}m^{9/2}c^2} + \ldots ,$$

$$P(0) = \frac{(mc^2)^{5/2}}{96\eta\mu_o^{3/2}} \left[x_F(2x_F^2-3)\left(1+\frac{1}{2}x_F^2-\frac{1}{8}x_F^4+\frac{1}{16}x_F^6+\ldots\right)\right.$$

$$\left.+3\left(x_F-\frac{1}{6}x_F^3+\frac{3}{40}x_F^5-\frac{5}{112}x_F^7+\ldots\right)\right]$$

$$= \frac{c\hbar^4(3\pi^2 n)^{4/3}}{15\eta\mu_o^{3/2}m^{3/2}} - \frac{3\hbar^7\pi^4 n^2}{56c\eta\mu_o^{3/2}m^{7/2}} + \ldots .$$

The first terms in the expansions of $E_t(0)$ and $P(0)$ are the values obtained in the case of the Schrödinger gas obtained earlier.

CHAPTER IV

APPLICATION TO A WHITE DWARF IN AN EXTERNAL MAGNETIC FIELD

In recent years it has been observed that white dwarfs[21] and pulsars[22,23] are accompanied by intense internal[10,24] magnetic fields of the order of 10^{10}-10^{13} Gauss. Although this field is generated internally,[25] it is still interesting to study the structure of white dwarfs in uniform external magnetic fields. This yields the general properties and characteristics of magnetic white dwarfs.[26] Here we use formulations developed in Appendix E for white dwarfs without magnetic fields.

A. Equation of Equilibrium

The total kinetic energy of the Dirac electron gas in a uniform magnetic field was given previously as

$$E = \frac{\mu_o N}{\eta n} B^{5/2} \left(\frac{2}{a} \Sigma_{-1/2}(a;s_F) - \frac{4}{a} \eta n B^{-3/2}\right.$$
$$\left. - \frac{\sqrt{s_F(1+as_F)}}{a} - a^{-3/2} \sinh^{-1} \sqrt{as_F}\right) . \tag{4.1}$$

The chemical potential [Eq. (3.7')] and the pressure [Eq. (3.16)] are

$$\mu = \varepsilon_F = mc^2(\sqrt{1+as_F} - 1) \tag{4.2}$$

and

$$P(B) = -\frac{\partial E}{\partial V} = \frac{4\mu_o nB}{a}\sqrt{1+as_F} - \frac{\mu_o B^{5/2}}{\eta}$$

$$[\frac{2}{a}\Sigma_{-1/2} - \frac{1}{a}\sqrt{s_F(1+as_F)} - a^{-3/2}\sinh^{-1}\sqrt{as_F}] \ . \tag{4.3}$$

Combining Eq. (4.2) with Eqs. (D.4) and (D.5), we arrive at

$$\frac{d^2}{dx^2}\sqrt{1+as_F} + \frac{2}{x}\frac{d}{dx}\sqrt{1+as_F} = -\frac{R^2}{b^2}[2\Lambda_{1/2}(s_F)-\sqrt{s_F}] \ ,$$

$$s_F(1) = \dot{s}_F(0) = 0 \ , \quad 0 \le x \le 1 \ , \tag{4.4}$$

where

$$b^2 = \frac{mc^2}{2\pi G m_o^2 B^{3/2}} \quad \text{and} \quad s_F(x)$$

is related to n(x) by

$$n(x) = \frac{B^{3/2}}{\eta}[\Lambda_{1/2}[s_F(x)] - \frac{1}{2}\sqrt{s_F(x)}]$$

from Eq. (3.6). Here $x = \gamma/R$, where R is the radius of white dwarf. To see the correctness of this result we take the zero field limit; i.e., we let $as_F \underset{B=0}{\sim} x_F^2$ and observe that

$$\nabla^2 \mu = -4\pi G\, m_o^2 n$$

with

$$\mu = mc^2(\sqrt{1+x_F^2} - 1)$$

as should be the case. Notice that s_F is a function of r through n(r) in Eq. (3.6). The mass-radius relation is

$$M[R] = -\frac{R^2\mu'(R)}{Gm_o} = -\frac{Rmc^2 a\dot{s}_F(1)}{2Gm_o} \,. \tag{4.5}$$

B. Ultra-Strong External Magnetic Field

The Eq. (4.4) is very complicated as it looks. We will not consider its general solution, but consider a special case of $B \geq (2\eta n_o)^{2/3}$, where $n_o \equiv n(0)$. In this extreme case all spins are down throughout the interior of the star and $0 < s_F(x) \leq 1$ with $s_F(x) = 4\eta^2 n^2(x) B^{-3}$. Then our Eq. (4.4) reduces to

$$\frac{d^2}{dx^2}\sqrt{1+as_F} + \frac{2}{x}\frac{d}{dx}\sqrt{1+as_F} = -\frac{R^2}{b^2}\, s_F \,,$$

$$s_F(1) = \dot{s}_F(0) = 0 \,, \quad 0 \leq x \leq 1 \,. \tag{4.6a}$$

By a change of variable, $\psi(x) = \frac{R}{b}(1+\sqrt{1+as_F})x$, Eq. (4.6a) becomes

$$\frac{d^2\psi}{dx^2} + \frac{R^2}{b^2\sqrt{a}}\sqrt{\psi^2 - \frac{2\psi xR}{b}} = 0 \ , \quad \psi(0) = 0, \ \psi(1) = \frac{2R}{b} \ . \tag{4.6b}$$

The mass-radius relation reduces to

$$M[R] = \frac{mc^2}{m_o G}\,[2R - b\dot{\psi}(1)]$$

The density n(x) is expressed in terms of $\psi(x)$ as

$$n(x) = \frac{bB^{3/2}\sqrt{\psi^2 - 2\psi xR/b}}{2\eta R\sqrt{a}\ x} = -\frac{B^{3/2}\ b^3\ddot{\psi}(x)}{2\eta xR^3} \ .$$

Furthermore Eq. (D.12) gives for this extreme case

$$\gamma_g[R] = \frac{1}{2} + \frac{1}{M^2}\left(\frac{m}{m_o}\right)^2 \frac{c^4}{4G^2}\left[b^2\int_0^1 [\dot{\psi}(x)]^2 dx - 4R^2\right] ,$$

which should be compared with Eq. (D.23). Now we show that this extreme case leads to absurd physical consequences. The critical mass M_o is defined as $M_o \equiv M[0]$, the limiting mass as R approaches zero. However, in this case we get $M_o = 0$ since $\lim_{R\to 0} \psi(x) = 0$. This is an absurd result. Secondly as $R\to 0$, $\gamma_g[R] \underset{R\to 0}{\sim} \frac{1}{2}$ and then Eq. (D.24) gives $\gamma_{th}[R] \underset{R\to 0}{\sim} \frac{1}{6}$. This implies $\lim_{R\to 0} E_{th}[R] < 0$ which is again absurd. Therefore we have proved the following theorem.

<u>THEOREM</u> White dwarfs can not be formed in an external magnetic field if $B \geq (2\eta n_o)^{2/3}$, where n_o is the central density of itinerant electrons.

In order to estimate the value of $B_c^{ex} = (2\eta n_o)^{2/3}$ we consider a white dwarf of $R/R_\odot = 0.0155$, $M/M_\odot = 0.4$, and $\rho_o = 1.072\times10^6$ g/cm^3 given in Table 3 of Appendix D. We find $B_c^{ex} = 1.79\times10^{15}$ Gauss a somewhat non-physical value. However, there exists a famous instability theorem by Chandrasekhar and Fermi for the magnetic stars accompanying the internal magnetic field B^{in}. This theorem states that the star is unstable if

$$|E_g| \leq \frac{1}{2\mu} \int (B^{in})^2\, dV = \frac{2\pi}{3\mu} (\bar{B}^{in})^2 R^3 ,$$

where E_g is the self-gravitational energy and $\mu = 4\pi\times10^{-7}$ henry/m. To find the critical magnetic field $\bar{B}_c^{in}$ of the magnetic white dwarfs above which the stars are unstable, we employ the expression of the self-gravitational energy E_g given by Eq. (D.25), $|E_g| = \gamma_g \frac{GM^2}{R}$, where $\frac{6}{7} < \gamma_g < \frac{3}{2}$. This yields $\bar{B}_c^{in}$ as $\bar{B}_c^{in} = 2.60\times10^8 \sqrt{\gamma_g}\, \frac{M/M_\odot}{(R/R_\odot)^2}$ Gauss and if $\bar{B}^{in} \geq B_c^{in}$ then the magnetic white dwarf is unstable. This gives $\bar{B}_c^{in} = 4.3\times10^{11}$ Gauss for the above white dwarf model. This may be compared with the earlier critical external field $B_c^{ex} = 1.79\times10^{15}$ Gauss.

CHAPTER V

DISCUSSION

The analysis given in Chapter II for the Schrödinger electron gas can readily be generalized to the case of the Bloch electrons at $T = 0^{\circ}K$ in which the conduction band can be written as

$$E(\underset{\sim}{k}) = \frac{\hbar^2}{2}\left(\frac{k_x^2}{m_1} + \frac{k_y^2}{m_2} + \frac{k_z^2}{m_3}\right) ,$$

an ellipsoidal constant energy surface near the minimum point $\underset{\sim}{k} = \underset{\sim}{0}$. We first observe that the wave function $\psi(\underset{\sim}{r})$ is expandable in terms of the orthonormalized Wannier functions in the narrow band approximation as

$$\psi(\underset{\sim}{r}) = \left(\frac{V}{N}\right)^{1/2} \sum_j F(\underset{\sim}{R}_j)\, a(\underset{\sim}{r}-\underset{\sim}{R}_j)$$

with the normalization condition

$$\int_V |\psi|^2 \, dV = \frac{V}{N} \sum_j |F(\underset{\sim}{R}_j)|^2 = 1 .$$

Here the summation is carried over N unit cells in the crystal volume V. It can be shown that $F(\underset{\sim}{r})$ satisfies the Schrödinger-like equation

$$\sum_1^3 \frac{1}{2m_r}\left(i\hbar \frac{\partial}{\partial x_r} - \frac{e}{c} A_r\right)^2 F(\underset{\sim}{r}) = E\, F(\underset{\sim}{r})$$

within the effective mass approximation. Setting $F(\underset{\sim}{r}) = \phi(x)\exp(ik_y y + ik_z z)$, then $\phi(x)$ satisfies the equation

$$\frac{d^2\phi}{dx'^2} + \frac{2m_1}{\hbar^2}\left(E'^2 - \frac{1}{2} m_1\omega_c^2 x'^2\right)\phi = 0 ,$$

where $E' = E - \hbar^2 k_z^2/2m_3$, $x' = x + c\hbar k_y/eB = \chi+\chi_o$, $\omega_c = eB/m_c c$, and $m_c = (m_1 m_2)^{1/2}$. This is the equation of the one-dimensional harmonic oscillator of mass m_1 and angular frequency ω_c. Consequently the energy eigenvalues read

$$E_n(k_z,\sigma) = (n+\tfrac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m_3} + \sigma\mu_o B$$

and the eigenfunction $F_{n,k_y,k_z}(\underset{\sim}{r})$ is the same as the earlier wavefunction $\psi_{n,k_y,k_z}(\underset{\sim}{r})$ for a free electron with α given by $(m_1\omega_c/\hbar)^{1/2}$ for the Bloch electrons, as was anticipated. The same degeneracy $g = 2eBL_1L_2/hc$ occurs. The rest of the analysis follows easily in exactly the same way as for free electrons at $T = 0^oK$. Here one notices that the constant A in the density of states should read $(eBV/h^2c)(2m_3/\hbar\omega_c)^{1/2}$. One also derives the same equation as Eq. (2.3) in which η is equal to $(\pi^4c^3\hbar^3mc/2e^3m_3)^{1/2}$. Furthermore the previous theorem still holds for the Bloch electrons at $T = 0^oK$ with this new value of η. We state without derivation that all the results obtained for free electrons hold for the Bloch electrons with μ_o

replaced by $e\hbar/2m_c c$ together with the new parameters discussed. Note that

$$\varepsilon_F(0) = \left(\frac{9\pi^4\hbar^6}{8m_1m_2m_3}\right)^{1/3} n^{2/3}$$

and

$$\chi_P = \left(\frac{3e^6m_3n}{64\pi^4c^6m_1^2m_2^2}\right)^{1/3} V .$$

For the spherical effective mass m^* the results are obtained trivially since $m_1=m_2=m_3=m^*$. In Appendix C, we use the same techniques to study the surface magnetic properties of a Bloch electron gas in opposing electric and magnetic fields at $T = 0^oK$.

Next we consider the system at $T \neq 0^oK$. For these non-zero temperatures one can rigorously prove the non-existence of the singularities in χ_t due to the smoothing factor $n(E) = 1/(e^{(E-\mu)/kT}+1)$. Here one finds a powerful method of finding the Helmholtz free energy F of the system through the grand partition function Ξ. We have

$$F = \mu N - kT \log \Xi,\ M_t = -\frac{\partial F}{\partial B},\ \chi_t = \frac{\partial M_t}{\partial B},$$

where

$$\log \Xi = \sum_{n,k_z,\sigma} g \log[1+ze^{-\beta E_n(k_z,\sigma)}]$$

$$= -\frac{eBV}{c\hbar^2}\left(\frac{m}{2\pi^3\beta}\right)^{1/2} \sum_1^\infty (-)^k\ k^{-3/2}z^k \coth(\beta\mu_o Bk).$$

Here

$$\beta = \frac{1}{kT} , \quad g = \frac{2eBL_1L_2}{hc} ,$$

the degeneracy, and $z = e^{\beta\mu}$, the fugacity.

The chemical potential $\mu(B)$ is to be obtained from the equation

$$N = z \frac{\partial}{\partial z} \log \Xi = - \frac{eBV}{c\hbar^2} \left(\frac{m}{2\pi^3\beta}\right)^{1/2} \sum_1^{\infty} (-)^k \, k^{1/2} \, z^k$$

$$\coth(\beta\mu_0 Bk) .$$

At this stage the use of the EM formula or Poissons summation formula is better justified because there are no singularities anywhere, in which case one is led to the oscillatory term.[6] We reproduce these formulas below

(EM formula)[28]

$$\sum_0^N f(n) = \int_0^N f(x)\,dx + \frac{1}{2} [f(0)+f(N)] + \sum_1^{m-1} \frac{B_{2n}}{(2n)!}$$

$$[f^{(2n-1)}(N) - f^{(2n-1)}(0)] + R_m(N) ,$$

where

$$F_m(N) = \frac{B_{2m}}{(2m)!} [f^{(2m-1)}(N) - f^{(2m-1)}(0)]$$

$$- \int_0^N \frac{B_{2m}(<x>)}{(2m)!} f^{(2m)}(x) \, dx ,$$

B_n and $B_n(x)$ are the Bernolli number and function, respectively.

(Poisson summation formula)[7]

$$\sum_{n_1}^{n_2} f(n+\tfrac{1}{2}) = \sum_{-\infty}^{\infty} (-)^P \int_{n_1}^{n_2+1} f(x) e^{2\pi i Px}\, dx$$

$$= \int_{n_1}^{n_2+1} f(x)\,dx + 2 \sum_{1}^{\infty} (-)^s \int_{n_1}^{n_2+1} f(x)\ \cos(2\pi sx)\,dx \ .$$

Finally, for the Bloch electrons at $T \neq 0^oK$, the inequality $\mu_o B \neq \frac{1}{2} \hbar\omega_c$ immediately affects the value of $\log \Xi$ such that

$$\log \Xi = - \frac{eBV}{c\hbar^2} \left(\frac{m_3}{2\pi^3\beta}\right)^{1/2} \sum_{1}^{\infty} (-)^k\ k^{-3/2}\ z^k$$

$$\frac{\cosh(\beta\mu_o Bk)}{\sinh(\beta\hbar\omega_c k/2)} \ .$$

This makes any subsequent analysis somewhat involved.

We further note that the interacting electron gas (or Bloch electrons) of the critical density n_c, below which the system is ferromagnetic, if it exists at all,[29] corresponds to the noninteracting electron gas with an effective external $\underset{\sim}{B}$ field of strength $B_c = (2\eta n_c)^{2/3}$. This is in the sense that they both have the same values of M_s from our theorem. (In the Hartree-Fock approximation with unscreened Coulomb interaction,[30]

$r_s = (2\pi/5)(9\pi/4)^{1/3}(2^{1/3}+1)a_o$ where $1/n_c = \frac{4}{3}\pi r_s^3$ and this gives $n_c = 9.952\times10^{21}$ cm^{-3} as the critical density for the full magnetization).

As regards to the Dirac Bloch[31,32] electrons on the ellipsoidal conduction band, we merely state that this conduction band will be written as

$$E(\underset{\sim}{k}) = mc^2 \left[1 + \frac{\hbar^2}{c^2}\left(\frac{k_x^2}{m_1^2} + \frac{k_y^2}{m_2^2} + \frac{k_z^2}{m_3^2}\right)\right]^{1/2} - mc^2 .$$

Compare this with the free particle dispersion relation

$$E(\underset{\sim}{k}) = mc^2 \left(1 + \frac{\hbar^2 k^2}{c^2 m^2}\right)^{1/2} - mc^2 = \sqrt{m^2c^4+c^2\hbar^2k^2} - mc^2 .$$

The Dirac spinor $\psi_s(\underset{\sim}{r})$ is expanded in terms of the orthonormalized Wannier spinors in the narrow band approximation as

$$\psi_s(\underset{\sim}{r}) = \left(\frac{V}{N}\right)^{1/2} \sum_j F(\underset{\sim}{R}_j) a_s(\underset{\sim}{r}-\underset{\sim}{R}_j) ,$$

where s = 1,2,3,4 is the spinor index. The Bloch spinor of n'th band $b_{ns}(\underset{\sim}{r})$ is related to the Wannier spinor $a_{ns}(\underset{\sim}{r}-\underset{\sim}{R}_j)$ through the transformation

$$b_{ns}(\underset{\sim}{k},\underset{\sim}{r}) = N^{-1/2} \sum_j \exp(i\underset{\sim}{k}\cdot\underset{\sim}{R}_j) a_{ns}(\underset{\sim}{r}-\underset{\sim}{R}_j) .$$

These two functions are orthonormalized as

$$\sum_{s} \int_{V} b_{ns}^{*}(\underset{\sim}{k},\underset{\sim}{r})\, b_{n's}(\underset{\sim}{k}',\underset{\sim}{r})\, d\underset{\sim}{r} = \delta_{nn'}\delta_{\underset{\sim}{k}\underset{\sim}{k}'}$$

and

$$\sum_{s} \int_{V} a_{ns}^{*}(\underset{\sim}{r}-\underset{\sim}{R}_{j})\, a_{n's}(\underset{\sim}{r}-\underset{\sim}{R}_{j'})\, d\underset{\sim}{r} = \delta_{nn'}\delta_{jj'} \ .$$

APPENDIX A

PROPERTIES OF THE FUNCTION $\Lambda_\mu(s)$

The function $\Lambda_\mu(s)$ is defined by

$$\Lambda_\mu(s) = \sum_{0}^{[s]} (s-n)^\mu \ , \quad s \geq 0 \ ,$$

where $[s]$ is the maximum integer not exceeding the value of s. This function is also written as

$$\Lambda_\mu(s) = \sum_{0}^{[s]} (\langle s\rangle+n) \quad , \quad s \geq 0 ,$$

where

$$\langle s\rangle = s-[s] \ .$$

The simple properties of $[s]$ and $\langle s\rangle$ are listed below.

(i) $[-s] = -[s] - 1, \quad \langle -s\rangle = 1 - \langle s\rangle \ ,$

(ii) $[s \pm N] = [s] \pm N \ , \quad \langle s \pm N\rangle = \langle s\rangle \ ,$

(iii) $[s_1+s_2] = [s_1]+[s_2] + [\langle s_1\rangle+\langle s_2\rangle] \ , \quad \langle s_1+s_2\rangle = \langle s_1\rangle+\langle s_2\rangle - [\langle s_1\rangle+\langle s_2\rangle],$

(iv) $[\langle s\rangle] = 0 \ , \quad \langle [s]\rangle = 0 \ ,$

(v) $[\varepsilon] = 0, \ [-\varepsilon] = -1, \quad \langle \varepsilon\rangle = \varepsilon, \ \langle -\varepsilon\rangle = -\varepsilon+1 \ ,$

(vi) $[\langle s\rangle+\varepsilon] = 0$, where ε is an infinitesimal positive.

From its definitions, we can derive the following properties of the function $\Lambda_\mu(s)$.

$$\Lambda_\mu(s+1) = \Lambda_\mu(s)+(s+1)^\mu \tag{A.1}$$

$\Lambda_\mu(s)$ is continuous if $\mu>0$ and has cusps at integer values $s=N$ for $0<\mu\leq 1$. Furthermore

$$\frac{d}{ds}\Lambda_\mu(s) = \mu\Lambda_{\mu-1}(s) \tag{A.2}$$

for all s if $1<\mu$ and for $s(\neq N)$ if $0<\mu\leq 1$. Also

$$\Lambda_\mu(s) = s^\mu , \quad \text{if} \quad 0 \leq s < 1 , \tag{A.3}$$

$$\Lambda_\mu(0) = 0 , \quad \text{if} \quad \mu > 0 ,$$

and for $\mu < 0$,

$$\lim_{\varepsilon\to 0} \Lambda_\mu(N+\varepsilon) = \infty ,$$

$$\lim_{\varepsilon\to 0} \Lambda_\mu(N-\varepsilon) = 1 + 2^\mu + 3^\mu + \ldots + N^\mu . \tag{A.4}$$

Fig. 17 shows the functions $\Lambda_\mu(s)$ for $\mu = -\frac{1}{2}, \frac{1}{2}, 1$, and $\frac{3}{2}$. We also observe

$$\Lambda_{-1/2}(s) = \frac{1}{\langle s\rangle^{1/2}} + \frac{1}{(\langle s\rangle+1)^{1/2}} + \cdots + \frac{1}{(s-1)^{1/2}}$$

$$+ \frac{1}{s^{1/2}} + \left[\frac{1}{(s+1)^{1/2}} + \frac{1}{(s+2)^{1/2}} + \cdots \right]$$

$$- \left[\frac{1}{(s+1)^{1/2}} + \frac{1}{(s+2)^{1/2}} + \cdots \right]$$

$$= \lim_{N\to\infty} \sum_{0}^{N} \{ (\langle s\rangle+n)^{-1/2} - (s+1+n)^{-1/2} \}$$

$$= \lim_{N\to\infty} \{ \sum_{0}^{N} (\langle s\rangle+n)^{-1/2} - 2N^{1/2} \} - \lim_{N\to\infty} \{ \sum_{0}^{N} (s+1+n)^{-1/2} - 2N^{1/2} \}$$

$$= \zeta(\tfrac{1}{2},\langle s\rangle) - \zeta(\tfrac{1}{2},s+1) \quad ,$$

where

$$\zeta(\nu,s) = \lim_{N\to\infty} \{ \sum_{0}^{N} (n+s)^{-\nu} - \frac{N^{1-\nu}}{1-\nu} \}.^{33}$$

APPENDIX B

ONSAGER'S FLUX QUANTIZATION

Historically Onsager's flux quantization was derived semiclassically using the Sommerfeld quantization rule of old quantum theory.[34] Although there exists a quantum mechanical[35] derivation of the flux quantization this does not take account of the persistent interaction of spin with the external magnetic field even if an electron is at rest. Here we make a logical derivation with inclusion of spin using Dirac theory.

We first realize that the notion of "electron orbit" has its meaning only in the classical domain. Therefore we can naturally employ the classical result. The rotational energy E_{rot} in classical physics is related to the radius of its orbit by $R^2 = 2E_{rot}^{cla}/m\omega_c^2$, where $\omega_c = eB/mc$ is the cyclotron frequency. In quantum mechanics this may be written as

$$\langle R^2 \rangle = \frac{2E_{rot}^{QM}}{m\omega_c^2} \ .$$

Quantum mechanically, one can still talk about the radius R of the equilibrium circle around which a quadratic fluctuation[36] is superimposed due to the uncertainty in the location of the center of the orbit.[17] However, E_{rot}^{QM} is obtained from the earlier energy eigenvalue in Dirac theory as below.

$$E_{rot}^{QM} = \lim_{V/c<<1} [E_{n\sigma}(0) - \sigma\mu_o B] = (n+\frac{1}{2})\hbar\omega_c \ ,$$

where

$$E_{n\sigma}(k_z) = (m^2c^4+c^2\hbar^2k_z^2+\varepsilon^2(n+\frac{1}{2})+\frac{1}{2}\sigma\varepsilon^2)^{1/2} - mc^2$$

as in Eq. (3.1). Finally we arrive at the area quantization

$$A_n = <\pi R^2> = \frac{hc}{eB}(n+\frac{1}{2}) \ , \quad n = 0,1,2,\ldots \ .$$

Therefore the flux passing through the orbit is quantized due to A_n as

$$\Phi = \frac{hc}{e}(n+\frac{1}{2}) \ ,$$

the famous Onsager's flux quantization.

APPENDIX C

SURFACE MAGNETIC PROPERTIES OF A BLOCH ELECTRON GAS IN OPPOSING ELECTRIC AND MAGNETIC FIELDS AT $T = 0^{\circ}K$

In recent years considerable attention has been paid to the properties of electrons in opposing electric and magnetic fields in conjunction with Shubnikov-de Haas oscillations[11] in the two-dimensional electron gas of semiconductor surface inversion layers.[38-40] Here we focus on the Bloch electrons in the conduction band which can be written as an ellipsoidal constant energy surface

$$E(\underset{\sim}{k}) = \frac{\hbar^2}{2}\left(\frac{k_x^2}{m_1} + \frac{k_y^2}{m_2} + \frac{k_z^2}{m_3}\right)$$

near the minimum point $\underset{\sim}{K} = \underset{\sim}{0}$. Furthermore, the total number of electrons N is fixed regardless of their origin. The field configurations are $\underset{\sim}{B} = (0,0,B)$, $\underset{\sim}{\varepsilon} = (0,0,-\varepsilon)$, and $\underset{\sim}{A} = (0,Bx,0)$, where $-\infty < x,y < +\infty$, $-\infty < z \leq 0$. As the boundary condition on the potential we take $V(x,y,0) = \infty$.

We first notice that the wave function $\psi(\underset{\sim}{r})$ is expanded in terms of the orthonormalized Wannier functions in the narrow band approximation as

$$\psi(\underset{\sim}{r}) = \left(\frac{V}{N}\right)^{1/2} \sum_j F(\underset{\sim}{R}_j)\, a(\underset{\sim}{r}-\underset{\sim}{R}_j)$$

with

$$\int_V |\psi|^2 dV = \frac{V}{N} \sum_j |F(\underset{\sim}{R}_j)|^2 = 1 \ .$$

Here the summation is carried over N unit cells in the crystal volume V. Then it can be shown that $F(\underset{\sim}{r})$ satisfies the Schrödinger-like equation

$$\{\sum_1^3 \frac{1}{2m_r} (i\hbar \frac{\partial}{\partial x_r} - \frac{e}{c} A_r)^2 - e\varepsilon z\} F(\underset{\sim}{r}) = EF(\underset{\sim}{r}) \ .$$

Setting $F(\underset{\sim}{r}) = F(x,y)F(z)$ and $E = E_{xy} + E_z$ we immediately obtain

$$F(x,y) = (\frac{\alpha}{\pi^{1/2} 2^n n! L_2})^{1/2} H_n[\alpha(x+x_o)] \ e^{-\frac{1}{2}\alpha^2 (x+x_o)^2} e^{ik_y y}$$

with

$$\iint |F(x,y)|^2 dxdy = 1 \ ,$$

where

$$\alpha^2 = (m_1 \omega_c/\hbar)^{1/2} \ , \qquad \omega_c = \frac{eB}{m_c c} \ , \qquad x_o = \frac{c\hbar k_y}{eB} \ ,$$

and

$$m_c = (m_1 m_2)^{1/2} \ .$$

The energy eigenvalue is given by

$$E_{xy} = (n + \tfrac{1}{2})\hbar\omega_c \ , \quad n = 0,1,2,\ldots \ .$$

We also find

$$F''(z) + azF(z) + bF(z) = 0 ,$$

where

$$a = \frac{2m_3 e\varepsilon}{\hbar^2} \quad \text{and} \quad b = \frac{2m_3 E_z}{\hbar^2} .$$

This has a solution in terms of Airy function[41]

$$F_s(z) = C_s A_i[-(a^{1/3}z + b_s a^{-2/3})] = C_s A_i[-(a^{1/3}z + z_s)],$$

where the boundary condition $F_s(0) = 0$ has been employed in the last expression and C_s is the normalization constant. Here the values of z_s are obtained from $A_i(-z_s) = 0$ and equal[42]

$$z_s = f\left[\frac{3\pi(4s-1)}{8}\right] ,$$

where

$$f(x) = 1 + \frac{x^3}{3!} + \frac{1.4}{6!}x^6 + \frac{1.4.7}{9!}x^9 + \ldots .$$

Hence the eigenvalue E_z is found to be

$$(E_z)_s = \frac{\hbar^2 a^{2/3} z_s}{2m_3} , \quad s = 1,2,\ldots .$$

Furthermore it is easily observed that the absolute value of the wave function $|F_s(z)|$ has peaks at $z = \bar{z}_{s,r} = -a^{-1/3}(z_s - z_r')$, where $-z_r'$ are the zeroes of $A_i'(z)$; i.e., $A_i'(-z_r') = 0$ and

$$z_r' = g\left[\frac{3\pi(4r-3)}{8}\right]$$

with

$$g(x) = x + \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 + \ldots .$$

Table 1 lists the values of z_s and z_s' up to s = 10. Finally we arrive at

$$E_n(s,\sigma) = (n+\tfrac{1}{2})\hbar\omega_c + \frac{\hbar^2 a^{2/3}}{2m_3} z_s + \sigma\mu_o B$$

with the degeneracy

$$g_E = \frac{eBL_1L_2}{hc} \quad ,$$

and

$$F_{n,k_y,s}(\underset{\sim}{r}) = F_{n,k_y}(x,y)F_s(z).$$

Comparing these energy eigenvalues and states with those of the electron in the $\underset{\sim}{B}$ field alone, i.e.

$$E_n(k_z,\sigma) = (n+\tfrac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m_3} + \sigma\mu_o B$$

with

$$g_E = \frac{2eBL_1L_2}{hc}$$

and

$$F_{n,k_y,k_z}(\underset{\sim}{r}) = F_{n,k_y}(x,y)\ \frac{1}{\sqrt{L_3}}\ e^{ik_z z}$$

we immediately realize that the translational degree of freedom along the $\underset{\sim}{B}$ field is lost and surface bound states are formed by the external $\underset{\sim}{\varepsilon}$ field. Through the expressions for energy eigenvalues and eigenstates we readily see the reason for the reduction in the degeneracy of the present energy level by one-half.

Here it is not possible to define any simple density of states D(E) as in a system with the $\underset{\sim}{B}$ field alone owing to the nature of totally bound states. Hence we confine ourselves to the case of $\hbar\omega_c << \hbar^2 a^{2/3}/2m_3$ or $B <<$ $m_c c\varepsilon^{2/3}/(2m_3 e\hbar)^{1/3}$. We now assume that only the lowest electric sub-level s = 1 is occupied. Fig. 18 shows $F_1(z)$ for the electric field strength ε = 3 Volt/10^{-5} cm. We obtain

$$N(E) = g_E\{[\frac{1}{\hbar\omega_c}\ (E-Az_1 - \frac{1}{2}\ \hbar\omega_c)]+1\} = g_E(n_{max}+1),$$

TABLE 1

Values of z_s and z_s'

s	z_s	z_s'
1	2.33810741	1.01879297
2	4.08794944	3.24819758
3	5.52055983	4.82009921
4	6.78670809	6.16330736
5	7.94413359	7.37217726
6	9.02265085	8.48848673
7	10.04017434	9.53544905
8	11.00852430	10.52766040
9	11.93601556	11.47505663
10	12.82877675	12.38478837

where

$$n_{max} = [\frac{E}{\hbar\omega_c} - \frac{1}{2} - \frac{Az_1}{\hbar\omega_c}]$$

and

$$A = \frac{\hbar^2 a^{2/3}}{2m_3} \quad .$$

Defining a new dimensionless variable s by

$$s = \frac{E}{\hbar\omega_c} - \frac{Az_1}{\hbar\omega_c} \quad ,$$

we write that

$$N(E) = g_E\{[s]+1\} \quad \text{and} \quad D(E) = \frac{dN}{dE} = \frac{g_E}{\hbar\omega_c}\frac{d[s]}{ds} \quad .$$

Next we observe that

$$N_\uparrow = \int_{\frac{\hbar\omega_c}{2}+\mu_o B+Az_1}^{\epsilon_F} D(E-\mu_o B)\,dE = g_E\int_0^{[s_F-1-Az_1/\hbar\omega_c]} d[s]$$

$$= g_E[s_F-1 - \frac{Az_1}{\hbar\omega_c}] \ ,$$

$$N_\downarrow = \int_{\frac{\hbar\omega_c}{2}-\mu_o B+Az_1}^{\epsilon_F} D(E+\mu_o B)\,dE = g_E\int_0^{[s_F-\frac{Az_1}{\hbar\omega_c}]} d[s]$$

$$= g_E[s_F - \frac{Az_1}{\hbar\omega_c}] \ ,$$

where $s_F = \epsilon_F/\hbar\omega_c$. Consequently we arrive at

$$N = N_\uparrow + N_\downarrow = g_E\{2[s_F - \frac{Az_1}{\hbar\omega_c}] - 1\}$$

or

$$\frac{hcn_s}{2eB} + \frac{1}{2} = [s_F - \frac{Az_1}{\hbar\omega_c}] \ ,$$

where

$$n_s = \frac{N}{L_1 L_2} \quad . \tag{C.1}$$

As is evident, the Fermi energy $\epsilon_F = \hbar\omega_c\ s_F$ is not well defined in our system. However, this is quite natural since the quantum states are effectively bound states. This situation is similar to trying to find ϵ_F for the orbital electrons in a hydrogen-like atom. Nevertheless, this fact does not affect any of the thermodynamic properties of the electron gas as we see below.

The spin magnetization and susceptibility M_s and χ_s are now given by

$$M_s = \mu_o(N_\downarrow - N_\uparrow) = \frac{e^2 BL_1L_2}{4\pi mc^2} \ , \tag{C.2}$$

$$\chi_s = \frac{\partial M_s}{\partial B} = \frac{e^2 L_1 L_2}{4\pi mc^2} \ . \tag{C.3}$$

To obtain the total energy E_t we first calculate

$$E_{\uparrow} = \int_{\frac{\hbar\omega_c}{2}+\mu_o B+Az_1}^{\varepsilon_F} ED(E-\mu_o B)\,dE = \hbar\omega_c\, g_E \int_0^{[s_F-1-\frac{Az_1}{\hbar\omega_c}]} s\,d[s] + (\hbar\omega_c + Az_1)\, g_E \int_0^{[s_F-1-\frac{Az_1}{\hbar\omega_c}]} d[s] \;,$$

$$E_{\downarrow} = \int_{\frac{\hbar\omega_c}{2}-\mu_o B+Az_1}^{\varepsilon_F} ED(E+\mu_o B)\,dE = \hbar\omega_c\, g_E \int_0^{[s_F-\frac{Az_1}{\hbar\omega_c}]} s\,d[s] + Az_1\, g_E \int_0^{[s_F-\frac{Az_1}{\hbar\omega_c}]} d[s] \;.$$

These are found to be

$$E_{\uparrow} = \hbar\omega_c\, g_E\{[s_F - \frac{Az_1}{\hbar\omega_c}] - 1\}\{\tfrac{1}{2}[s_F - \frac{Az_1}{\hbar\omega_c}] + 1\}$$

$$+ Az_1 g_E\{[s_F - \frac{Az_1}{\hbar\omega_c}] - 1\}$$

and

$$E_{\downarrow} = \frac{1}{2}\hbar\omega_c\, g_E[s_F - \frac{Az_1}{\hbar\omega_c}]\,\{[s_F - \frac{Az_1}{\hbar\omega_c}] + 1\}$$

$$+ Az_1 g_E[s_F - \frac{Az_1}{\hbar\omega_c}] \ .$$

Consequently the total energy becomes

$$E_t = E_{\uparrow}+E_{\downarrow} = 2Az_1 g_E[s_F - \frac{Az_1}{\hbar\omega_c}] - Az_1 g_E$$

$$+ \hbar\omega_c g_E \quad [s_F - \frac{Az_1}{\hbar\omega_c}]^2 + \{[s_F - \frac{Az_1}{\hbar\omega_c}] - 1\}.$$

Now using Eq. (C.1) this simplifies to

$$E_t = (Az_1 + \hbar\omega_c + \frac{\pi\hbar^2 n_s}{2m_c})N - \frac{e^2B^2L_1L_2}{8\pi m_c c^2} \ . \qquad \text{(C.4)}$$

This in turn gives

$$M_t = -\frac{\partial E_t}{\partial B} = \frac{e^2BL_1L_2}{4\pi m_c c^2} - \frac{e\hbar N}{m_c c} \ , \qquad \text{(C.5)}$$

$$\chi_t = \frac{\partial M_t}{\partial B} = \frac{e^2L_1L_2}{4\pi m_c c^2} \ , \qquad \text{(C.6)}$$

$$\chi_{orb} = \chi_t - \chi_s = \frac{e^2L_1L_2}{4\pi c^2}\ (\frac{1}{m_c} - \frac{1}{m}) \ , \qquad \text{(C.7)}$$

where

$$\chi_s = \frac{e^2L_1L_2}{4\pi mc^2} \ .$$

The expressions Eqs. (C.2) - (C.7) clearly indicate surface and volume contributions. Furthermore, the magnetic susceptibilities contain pure surface contributions since the electrons form bound states only near the surface and show no singularities even at $T = 0^{o}K$. This last property is due to the nature of bound states and non-existence of the free particle-like behavior. Notice that the total susceptibility χ_t in Eq. (C.6) may be used to determine the cyclotron mass m_c experimentally.

APPENDIX D

A WHITE DWARF WITHOUT AN EXTERNAL MAGNETIC FIELD†

1. Equilibrium Equation

The idealized model for cold dense stars like white dwarfs consists of a gas of N noninteracting electrons in their ground state moving in a background of N/Z motionless nuclei with Z protons which provide the gravitational attraction to hold the entire system together against the tremendous pressure created by the itinerant electrons. In other words, the system under study is essentially an inhomogeneous electron gas of spherical density distribution under an external gravitational field produced by the background nuclei. The system is further taken to be nonrotating. This problem was advanced as long ago as 40 years by Chandrasekhar[43] and parameterized in terms of the central density ρ_o. However, in our present discussion, we use the parameter R, the radius of the star, instead of ρ_o, through which we are able to obtain the exact total binding and self-gravitational energy. The advantage of using R over using ρ_o, beyond the resulting simplicity, will be demonstrated throughout the formalisms.

The total kinetic energy of the Dirac electron gas will be

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†All the symbols used in this Appendix are listed in Table 2.

Table 2

List of Symbols Used

Symbols	Meaning
ε_F	Fermi energy of an electron gas
m	mass of an electron
n	number density of an electron gas
α	$\frac{\hbar(3\pi^2)^{1/3}}{mc}$
x_F	$\alpha n^{1/3}(=[(\frac{AS}{Rx})^2 - 1]^{1/2})$
G	universal gravitational constant
A	$(\frac{\alpha^3 mc^2}{4\pi G m_o^2})^{1/2}$
m_o	$\frac{m_n}{Z} + m \simeq \frac{m_n}{Z}$, m_n: mass of the nucleus
M	total mass of the system ($=Nm_o$)
$\tilde{S}(x)$	$(\frac{2R}{A})^3[S(x) - \frac{R}{A}x]$

$$E = 2 \sum_{|\underset{\sim}{P}| \le P_f} [(m^2c^4 + P^2c^2)^{1/2} - mc^2] \tag{D.1}$$

$$= \frac{m^4c^5V}{8\pi^2\hbar^3}[x_F(1 + 2x_F^2)(1 + x_F^2)^{1/2} - \sinh^{-1}x_F - \frac{8}{3}x_F^3]$$

the same as Eq. (3.30). Accordingly the pressure and chemical potential read

$$P = \frac{mc^2}{8\alpha^3}[x_F(2x_F^2 - 3)(1 + x_F^2)^{1/2} + 3\sinh^{-1}x_F] \tag{D.2}$$

$$\mu = \varepsilon_F = mc^2[(1 + x_F^2)^{1/2} - 1]. \tag{D.3}$$

Then one immediately writes the following.

$$\mu + m_o\phi = -\frac{GMm_o}{R} \tag{D.4}$$

$$\nabla^2\phi = 4\pi G\rho. \tag{D.5}$$

Here $\rho = nm_o$ is the density of the system. Combining Eqs. (D.4) and (D.5) we obtain

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\mu}{dr}) = -4\pi Gm_o^2n. \tag{D.6}$$

This is again combined with Eq. (D.3) to yield

$$\frac{d^2\mu}{dr^2} + \frac{2}{r}\frac{d\mu}{dr} + \frac{4\pi Gm_o^2}{\alpha^3}[(1 + \frac{\mu}{mc^2})^2 - 1]^{3/2} = 0,$$

$$\mu(R) = \mu'(0) = 0. \tag{D.7}$$

Now we define new dimensionless variables $x = \frac{r}{R}$ and $S(x) = \frac{R}{A}(1 + \frac{\mu}{mc^2})x$ so now Eq. (D.7) becomes

$$x^2 \frac{d^2S}{dx^2} = -(S^2 - \frac{R^2}{A^2}x^2)^{3/2}, \; S(0)=0, S(1)=\frac{R}{A}, 0 \leq x \leq 1. \quad \text{(D.8a)}$$

This is the desired exact equation of equilibrium for a cold dense star of radius R for our subsequent discussions. Equation (D.8a) is at least a simplified version of the famous Chandrasekhar equation of equilibrium which reads

$$\frac{d^2\phi}{d\eta^2} + \frac{2}{\eta}\frac{d\phi}{d\eta} = -(\phi^2 - \frac{1}{\zeta^2})^{3/2}, \; \phi(0)=1, \phi'(0)=0, 0 \leq \eta < \infty,$$

where $\phi = [\frac{1 + \alpha^2 n^{2/3}(r)}{1 + \alpha^2 n^{2/3}(0)}]^{1/2}$, $\zeta_o = [1 + \alpha^2 n^{2/3}(0)]^{1/2}$, and $\eta = \zeta_o \frac{r}{A}$ in our notation. We also notice that

$$n(x) = \frac{1}{\alpha^3}[(\frac{AS}{Rx})^2 - 1]^{3/2} \quad \text{(D.9a)}$$

or equivalently

$$n(x) = -\frac{A^3}{\alpha^3 R^3 x}\frac{d^2S}{dx^2}.$$

Therefore we see that $\frac{d^2S}{dx^2} \leq 0$ and the function $\dot{S}(x)$ is a monotonically decreasing function of x, where the equality in the above relation holds on the surface only. The ratio between the central density and the average one reads

$$\frac{\rho_o}{\bar{\rho}} = \frac{[(\lim_{x\to 0} \frac{S}{x})^2 - \frac{R^2}{A^2}]^{3/2}}{3[\frac{R}{A} - \dot{S}(1)]}, \quad \text{(D.10)}$$

where Eq. (D.20) has been utilized. Next we investigate two limiting cases.

Case [1] $x_F >> 1$: Ultrarelativistic gas

Writing Eq. (D.9a) as

$$n(x) = \frac{1}{\alpha^3}\left(\frac{AS}{Rx}\right)^3\left[1 - \left(\frac{Rx}{AS}\right)^2\right]^{3/2}$$
$$= \frac{1}{\alpha^3}\left(\frac{AS}{Rx}\right)^3\left[1 - \frac{3}{2}\left(\frac{Rx}{AS}\right)^2 + \frac{3}{8}\left(\frac{Rx}{AS}\right)^4 - \ldots.\right] \tag{D.9b}$$

then this limit corresponds to $\frac{Rx}{AS} << 1$. Consequently Eq. (D.8a) can be expanded as

$$x^2 \frac{d^2S}{dx^2} = -S^3\left[1 - \left(\frac{Rx}{AS}\right)^2\right]^{3/2}$$
$$= -S^3\left[1 - \frac{3}{2}\left(\frac{Rx}{AS}\right)^2 + \frac{3}{8}\left(\frac{Rx}{AS}\right)^4 - \ldots.\right]. \tag{D.8b}$$

Taking the first term in the expansion we obtain

$$x^2 \frac{d^2S_R}{dx^2} = - S_R^3, \quad S_R(0) = S_R(1) = 0, \quad 0 \leq x \leq 1, \tag{D.11}$$

in which $\lim_{x_R>>1} S(x) = S_R(x)$. A calculation shows that $\dot{S}_R(1) = -2.0182$ and $\lim_{x\to 0} \frac{S_R}{x} = 6.897$. The density in this limit becomes $n(x) = \frac{1}{\alpha^3}\left(\frac{AS_R}{Rx}\right)^3$. Alternatively one can obtain Eq. (D.11) by noticing $\mu = mc^2\alpha n^{1/3}$ in this limit and $\mu = mc^2 \frac{AS_R}{Rx}$ in Eq. (D.6). It is also interesting to realize that the function S_R is the one which minimizes the following variational functional $I_R[f]$ with the same boundary conditions:

$$I_R[f] = \int_0^1 \left(\frac{1}{2x^2} \dot{f}^4 - f^2\right)dx, \quad f(0)=f(1)=0, \quad \dot{f}=\frac{df}{dx}. \tag{D.12}$$

Case [2] $x_F << 1$: Nonrelativistic gas

We define a new function $\tilde{S}(x)$ by

$$S(x) = \frac{R}{A}[x + 2(\frac{A}{2R})^4 \tilde{S}(x)]$$

and we write Eq. (D.9a) as

$$\begin{aligned} n(x) &= \frac{8}{\alpha^3}(\frac{A}{2R})^6 (\frac{\tilde{S}}{x})^{3/2}[1 + (\frac{A}{2R})^4 \frac{\tilde{S}}{x}]^{3/2} \\ &= \frac{8}{\alpha^3}(\frac{A}{2R})^6 (\frac{\tilde{S}}{x})^{3/2}[1 + \frac{3}{2}(\frac{A}{2R})^4 \frac{\tilde{S}}{x} + \frac{3}{8}(\frac{A}{2R})^8 (\frac{\tilde{S}}{x})^2 + \ldots.]. \end{aligned} \tag{D.9c}$$

Then this limit corresponds to $(\frac{A}{2R})^4 \frac{\tilde{S}}{x} << 1$ and therefore Eq. (D.8a) is to be expanded as

$$\begin{aligned} x^{1/2} \frac{d^2\tilde{S}}{dx^2} &= -\tilde{S}^{3/2}[1 + (\frac{A}{2R})^4 \frac{\tilde{S}}{x}]^{3/2} \\ &= -\tilde{S}^{3/2}[1 + \frac{3}{2}(\frac{A}{2R})^4 \frac{\tilde{S}}{x} + \frac{3}{8}(\frac{A}{2R})^8 (\frac{\tilde{S}}{x})^2 + \ldots.], \\ & S(0) = S(1) = 0. \end{aligned} \tag{D.8c}$$

Again taking the first term we obtain

$$x^{1/2} \frac{d^2\tilde{S}_N}{dx^2} = -\tilde{S}_N^{3/2}, \; \tilde{S}_N(0) = \tilde{S}_N(1) = 0, \; 0 \le x \le 1, \tag{D.13}$$

in which $\lim_{x_F<<1} \tilde{S}(x) = \tilde{S}_N(x)$ and therefore $\lim_{x_F<<1} S(x) = \frac{R}{A}[x + 2(\frac{A}{2R})^4 \tilde{S}_N]$. A calculation shows that $\dot{\tilde{S}}_N(1) = -132.4$ and $\lim_{x\to 0} \frac{\tilde{S}_N}{x} = 178.2$. Alternatively one obtains Eq. (D.13) by observing $\mu = \frac{1}{2}mc^2\alpha^2 n^{2/3}$ and using the transformation $\mu = 2mc^2(\frac{A}{2R})^4 \frac{\tilde{S}_N}{x}$ in Eq. (D.6). The density in this limit becomes $n(x) = \frac{8}{\alpha^3}[(\frac{A}{2R})^4 \frac{\tilde{S}_N}{x}]^{3/2}$. The corresponding functional to $\tilde{S}_N$ will be

$$I_N[f] = \int_0^1 [\tfrac{4}{5}x^{-1/2}f^{5/2} - \dot{f}^2]dx, \; f(0) = f(1) = 0. \quad \text{(D.14)}$$

It should be emphasized that the three Eqs. (D.8a), (D.8b), and (D.8c) are equivalent but written in different forms for different purposes. The expansions shown in Eqs. (D.8b) and (D.8c) intuitively suggest the iterative approximations converging to the exact solutions S(x) and $\tilde{S}(x)$ respectively. These are discussed in section 3. The equivalent equations to our Eq. (D.11) and (D.13) also appear in the discussions by Landau and Lifshitz.[44] However, their formulations neither lead to the exact Eq. (D.8a) nor its subsequent expansions for the two limiting cases. We also note that Eq. (D.11) is directly obtained from Eq. (D.8a) by taking the limit $R \to 0$. Therefore, S_R is the exact solution for a star of $R = 0$ and $\rho = \infty$, where the use of the Newtonian theory of gravitation is totally unjustified. For this reason this limit should not be taken too seriously as far as the use of the general theory of relativity is avoided. In the limit $R \to 0$ the mass becomes M_o, the critical mass above which the system is expected to undergo the so-called gravitational collapse.[45,46]

Next we obtain a new equilibrium condition. The self-gravitational energy is given by

$$E_g = \frac{1}{2}\int \rho\phi dV = -\frac{1}{2}\int \mu n dV - \frac{1}{2}\frac{GM^2}{R} \tag{D.15}$$

or alternatively

$$E_g = -3\int P dV. \tag{D.16}$$

Combining these two equations and using Eqs. (D.2) and (D.3) we arrive at

$$\int_0^R [2x_F^3(1+x_F^2)^{1/2}+4x_F^3-9x_F(1+x_F^2)^{1/2}+9\sinh^{-1}x_F]r^2dr = \alpha\frac{3GM^2}{\pi mc^2 R}. \tag{D.17}$$

Expressing x_F in terms of $S(x)$ and using Eq. (D.8a) it is not hard to show that

$$\int_0^1 \{2A^3\dot{S}^2 - 9R^3xS[(\frac{AS}{Rx})^2 - 1]^{1/2} - 3R^3\frac{x^2\dot{S} - xS}{[(\frac{AS}{Rx})^2 - 1]^{1/2}}\}dx = 4A^2\dot{S}(1)[A\dot{S}(1) - 2R], \tag{D.17)'$$

where $\dot{S} = \frac{dS}{dx}$. This is an another equation which $S(x)$ satisfies besides its non-linear differential equation (D.8a). In the limit as $R\to 0$ and $S\to S_R$ we immediately derive the formula

$$\int_0^1 \dot{S}_R^2 dx = 2[\dot{S}_R(1)]^2, \tag{D.18}$$

which will be used later. To derive a similar formula for the function $\tilde{S}_N(x)$ we transform $\tilde{S}$ into S in Eq. (D.17)'. One easily obtains after some calculations that

$$\int_0^1 [\dot{\tilde{S}}_N]^2 dx = \frac{5}{7}[\dot{\tilde{S}}_N(1)]^2 \tag{D.19}$$

or equivalently

$$\lim_{x_F<<1} \int_0^1 [\dot{S}]^2 dx = \frac{5}{448}(\frac{A}{R})^6 [\dot{\tilde{S}}_N(1)]^2 + \frac{R^2}{A^2} \tag{D.20}$$

which will be used in later discussions of a limiting case. We can also obtain the criteria for the validity of these two limiting cases. Since $\lim_{x\to 0} \frac{S_R}{x} = 6.897$ and $\lim_{x\to 0} \frac{\tilde{S}_N}{x} = 178.2$ it is trivial to show that $x_F >> 1$ for $R << 6.897A$ and $x_F << 1$ for $R >> \frac{A}{2}(178.2)^{1/4}$. However, within the star, $x_F >> 1$ belongs to the region near the center and the $x_F << 1$ limit corresponds to the region near the surface. Next we derive the mass-radius relation M[R] in terms of S(x).

Integrating Eq. (D.5) from 0 to R, we obtain

$$-\frac{GM}{R^2} = \frac{1}{m_o}\mu'(R) = \frac{mc^2\alpha^2}{2m_o}[1 + \alpha^2 n^{2/3})^{-1/2} \frac{dn^{2/3}}{dr}]_{r=R} \,. \tag{D.21a}$$

This is equal to

$$M[R] = \frac{mc^2}{m_o G}[R - A\dot{S}(1)]. \tag{D.22}$$

Some of the values of $\dot{S}(1)$ are tabulated in Table 3 for different radii R.

Finally we check our formula by taking the two limits

$$\lim_{x_F>>1} M[R] = -\frac{m}{m_o}\frac{c^2 A}{G}\dot{S}_R(1) \tag{D.23}$$

and

$$\lim_{x_F<<1} M[R] = -\frac{m}{m_o}\frac{c^2 A^4}{8G}\frac{\dot{\tilde{S}}_N(1)}{R^3}, \tag{D.24}$$

Table 3

Quantities of Interest for Eleven Different Radius Parameters R
Here we set $m_o = 2\ m_p$.

$M_\odot$:mass of the sun; $R_\odot$:radius of the sun; $A=3.8542\times10^8$ cm

$M/M_\odot$	$R/R_\odot$	$S'(1)$	$\lim_{x\to0}\frac{S(x)}{x}$	$\rho(0)=m_o n(0)$ $[g/cm^3]$	$X_F(0)$
0.22	0.0200	+3.2993	4.0330	2.454×10^5	0.4995
0.40	0.0155	+2.2348	3.6099	1.072×10^6	0.8166
0.50	0.0138	+1.7885	3.5154	1.950×10^6	0.9966
0.61	0.0123	+1.3631	3.5109	3.631×10^6	1.2261
0.74	0.0110	+0.9457	3.6293	7.080×10^6	1.5318
0.88	0.0093	+0.4483	3.7694	1.585×10^7	2.0041
1.08	0.0071	+0.2381	4.0233	5.250×10^7	2.9871
1.22	0.0055	-0.7217	4.4178	1.620×10^8	4.3489
1.33	0.0039	-1.1641	4.9647	6.760×10^8	7.0011
1.38	0.0030	-1.4042	5.7032	2.400×10^9	10.6736
1.44	0.0000	-2.0182		∞	∞

which are the well-known relations in these limits. The three curves (D.22) to (D.24) are shown in Fig. 19. The M[R] of $x_F >> 1$ limit does not contain R in it; however, this should be so due to the fact that this limit corresponds to R = 0 star. Therefore Eq. (D.23) is the critical mass of our system within the Newtonian theory of gravitation. For white dwarfs with $m_o = 2m_P$, where m_P is the mass of a proton, this gives $M_o = 1.438\ M_\odot$. The M_o for neutron stars* becomes 5.754 $M_\odot$ which is much larger than the more accurate result 0.76 $M_\odot$ based on the general theory of relativity calculated by Oppenheimer and Volkoff.[51] Next we employ Eq. (D.21a) to obtain better approximations than given by Eqs. (D.23) and (D.24).

Case [1] $x_F >> 1$

Equation (D.21a) is expanded as

$$-\frac{GM}{R^2} = \frac{mc^2\alpha}{2m_o}\left[n^{-1/3}\left(1 - \frac{1}{2}\alpha^{-2}n^{-2/3}+\ldots\right)\frac{dn^{2/3}}{dr}\right]_{r=R} . \qquad \text{(D.21b)}$$

Notice that taking just the first term leads to Eq. (D.23).

Case [2] $x_F << 1$

Equation (D.21a) is to be expanded in this limit as

$$-\frac{GM}{R^2} = \frac{mc^2\alpha^2}{2m_o}\left[\left(1 - \frac{1}{2}\alpha^2 n^{2/3}+\ldots\right)\frac{dn^{2/3}}{dr}\right]_{r=R} . \qquad \text{(D.21c)}$$

---

*This follows upon replacements of the electronic parameters with the neutron's, namely n:neutron density and m, m_o:neutron mass.

Again the first term leads to Eq. (D.24). The next approximations in two regions other than the two extreme limits (D.23) and (D.24) are obtained in section 3.

2. Exact Self-Gravitational and Binding Energies

Writing Eq. (D.15) as $E_g = -\gamma_g \frac{GM^2}{R}$, we obtain

$$\gamma_g[R] = \frac{1}{2} + \frac{1}{M^2}\left(\frac{m}{m_o}\right)^2 \frac{c^4 A}{2G^2}\left[A\int_0^1 [\dot{S}(x)]^2 dx - R\dot{S}(1)\right]. \quad (D.25)$$

This is expressed solely in terms of R when Eq. (D.22) is substituted into it. Next we obtain the thermal kinetic energy of the electron gas, $E_{th} = -\gamma_{th}\frac{GM^2}{R}$. Since $E_{th} = \int_0^R (\mu n - P)dV$ one easily obtains upon using Eqs. (D.15, 16)

$$\gamma_{th}[R] = 1 - \frac{5}{3}\gamma_g[R]. \quad (D.26)$$

Finally we arrive at $E_{tot} = E_g + E_{th} = -\gamma_{tot}\frac{GM^2}{R}$, where

$$\gamma_{tot}[R] = \gamma_g[R] + \gamma_{th}[R] = 1 - \frac{2}{3}\gamma_g[R]. \quad (D.27)$$

This gives the total binding energy of a cold dense star of radius R.

Next we recall the well-known and important formulas for the polytrope leading to the Lane-Emden equation of index n.[47] If $\mu \propto \rho^n$ or equivalently $P \propto \frac{n}{n+1}\frac{\rho^{n+1}}{m_o}$ then

$$\gamma_g = \frac{3n}{5n-1},\ \gamma_{th} = -\frac{1}{5n-1},\ \text{and}\ \gamma_{tot} = \frac{3n-1}{5n-1}.$$

Noticing that the ultrarelativistic region corresponds to $n = \frac{1}{3}$, whereas the nonrelativistic case corresponds to $n = \frac{2}{3}$, it is trivial to prove the following two limiting values of our exact formulas (D.25,26,27) using the previous Eqs. (D.18) and (D.20).

$$\lim_{x_F \gg 1} \gamma_g[R] = \gamma_g[0] = \frac{3}{2}; \quad \lim_{x_F \ll 1} \gamma_g[R] = \frac{6}{7}$$

$$\lim_{x_F \gg 1} \gamma_{th}[R] = -\frac{3}{2}; \quad \lim_{x_F \ll 1} \gamma_{th}[R] = -\frac{3}{7}$$

$$\lim_{x_F \gg 1} \gamma_{tot}[R] = 0; \quad \lim_{x_F \ll 1} \gamma_{tot}[R] = \frac{3}{7},$$

where Eqs. (D.23) and (D.24) have been used in the corresponding limits. The qualitative behaviors of γ_g, γ_{th} and γ_{tot} as functions of R are shown in Fig. 20.

Finally we calculate $I_R[S_R]$ and $I_N[\tilde{S}_N]$, the minimum values of I_R and I_N respectively. We obtain

$$I_R[S_R] = \int_0^1 \left(\frac{1}{2x^2} S_R^4 - \dot{S}_R^2\right) dx = -[\dot{S}_R(1)]^2$$

using Eqs. (D.11) and (D.18). Similarly,

$$I_N[\tilde{S}_N] = \int_0^1 \left(\frac{4}{5} x^{-1/2} \tilde{S}_N^{5/2} - \dot{\tilde{S}}_N^2\right) dx = -\frac{1}{7}[\dot{S}_N(1)]^2,$$

where Eqs. (D.13) and (D.19) have been utilized.

A recent analysis of this system on the basis of the general theory of relativity, including the rotation of the star, was given by Van Riper;[48] however, it should be noticed that this is only for a polytrope of the fixed index n. This is analogous to the formulations by Landau and Lifshitz[44] using the Newtonian theory of gravitation.

3. Iterative Approximations to the Exact Solutions $S(x)$ and $\tilde{S}(x)$

Case [1] $x_F >> 1$

Equation (D.8b) is rewritten as

$$x^2 \frac{d^2S}{dx^2} = -S^3 \sum_{o}^{\infty} (-1)^n c_n \left(\frac{Rx}{AS}\right)^{2n},$$

where

$$c_n = \frac{\frac{3}{2}(\frac{3}{2} - 1)\ldots(\frac{3}{2} - n + 1)}{n}.$$

We now define $S_R^{(k)}$ by the differential equation

$$x^2 \frac{d^2S_R^{(k)}}{dx^2} = -[S_R^{(k)}]^3 \sum_{o}^{k} (-1)^n c_n \left(\frac{Rx}{AS_R^{(k)}}\right)^{2n} \tag{D.28}$$

with $S_R^{(k)}(0) = 0$ and $S_R^{(k)}(1) = \frac{R}{A}$. Here $S_R^{(0)} \equiv S_R$ and $S_R^{(0)}(1) = 0$. Next we replace $S_R^{(k)}$ in the right-hand side of Eq. (D.28) by $S_R^{(k-1)}$ and obtain the approximate differential equation for $S_R^{(k)}$

$$\frac{d^2S_R^{(k)}}{dx^2} = \frac{d^2S_R^{(k-1)}}{dx^2} + W_R^{(k-1)}(x), \tag{D.29}$$

where $W_R^{(k)}(x) = (-1)^k c_{k+1} (\frac{R}{A})^{2(k+1)} x^{2k} [S_R^{(k)}]^{1-2k}$.

Integrating Eq. (D.29) from 0 to x we finally arrive at

$$S_R^{(k)}(x) = S_R^{(k-1)}(x) + \int_0^x \int_0^{x'} W_R^{(k-1)} dx''dx'$$
$$- x\int_0^1 \int_0^{x'} W_R^{(k-1)} dx''dx'. \quad \text{(D.30)}$$

The last term insures we have the correct boundary condition at $x = 1$. This scheme starts from the function $S_R(x)$ and then the next function $S_R^{(1)}$ should read

$$S_R^{(1)} = S_R + \frac{3}{2}\left(\frac{R}{A}\right)^2 \int_0^x \int_0^{x'} S_R dx''dx' - \frac{3}{2}\left(\frac{R}{A}\right)^2 x\int_0^1\int_0^{x'} S_R dx''dx'.$$

Since $R << A$ in this primary stage, the corrections to the function $S_R(x)$ are seen to be small enough to continue our iteration as R becomes larger. In this way one can improve the accuracy for a star ($R \neq 0$) in the ultrarelativistic region. Notice that we started from a star with $R = 0$ and the radius R appears naturally as we go on with the iteration. The employment of the exact equation of equilibrium (D.8a) is good for only one radius R. Accordingly we define $n^{(k)}(x)$ and $M^{(k)}[R]$ by

$$n^{(k)}(x) = \frac{1}{3}\left[\left(\frac{AS_R^{(k)}}{Rx}\right)^2 - 1\right]^{3/2} \quad \text{(D.31)}$$

and

$$M^{(k)}[R] = \frac{mc^2}{m_o G}[R - A\dot{S}_R^{(k)}(1)], \quad \text{(D.32)}$$

where

$$\dot{S}_R^{(k)} = \dot{S}_R^{(k-1)} + \int_0^x W^{(k-1)} dx'' - \int_0^1\int_0^{x'} W^{(k-1)} dx''dx'.$$

It is ovbious that this iteration is valid for $\frac{R}{A} << 1$. An exactly similar analysis applies to the case of $x_F << 1$ below.

Case [2] $x_F << 1$

Writing Eq. (D.8c) as

$$x^{1/2} \frac{d^2\tilde{S}}{dx^2} = -\tilde{S}^{3/2} \sum_0^\infty c_n (\frac{A}{2R})^{4n} (\frac{\tilde{S}}{x})^n$$

we define the function $\tilde{S}_N^{(k)}$ by the differential equation

$$x^{1/2} \frac{d^2\tilde{S}_N^{(k)}}{dx^2} = -[\tilde{S}_N^{(k)}]^{3/2} \sum_0^k c_n (\frac{A}{2R})^{4n} (\frac{\tilde{S}_N^{(k)}}{x})^n,$$

$$\tilde{S}_N^{(k)}(0) = \tilde{S}_N^{(k)}(1) = 0 \tag{D.33}$$

with $\tilde{S}_N^{(0)} \equiv \tilde{S}_N$. In the same way as in case [1] we obtain the approximate equation for $\tilde{S}_N^{(k)}$

$$\frac{d^2\tilde{S}_N^{(k)}}{dx^2} = \frac{d^2\tilde{S}_N^{(k-1)}}{dx^2} - W_N^{(k-1)}(x), \tag{D.34}$$

where

$$W_N^{(k)} = c_{k+1} (\frac{A}{2R})^{4(k+1)} \frac{[\tilde{S}_N^{(k)}]^{k+\frac{5}{2}}}{x^{k+\frac{3}{2}}} .$$

This gives the solution as

$$\tilde{S}_N^{(k)} = \tilde{S}_N^{(k-1)} - \int_0^x \int_0^{x'} W_N^{(k-1)} dx''dx' + x\int_0^1 \int_0^{x'} W_N^{(k-1)} dx''dx'. \tag{D.35}$$

This scheme again starts from the function $\tilde{S}_N(x)$ and the next function $\tilde{S}_N^{(1)}(x)$ should read

$$\tilde{S}_N^{(1)} = \tilde{S}_N - \frac{3}{2}(\frac{A}{2R})^4 \int_0^x \int_0^{x'} \frac{\tilde{S}_N^{5/2}}{(x'')^{3/2}} dx''dx'$$

$$+ \frac{3}{2}(\frac{A}{2R})^4 x\int_0^1 \int_0^{x'} \frac{\tilde{S}_N^{5/2}}{(x'')^{3/2}} dx''dx'.$$

In this way we can construct the approximation to the exact function $\tilde{S}(x)$ starting from $\tilde{S}_N(x)$ which is free from the parameter R. Consequently the approximate solution to Eq. (D.8a) in this limit, $S_N^{(k)}$, becomes

$$S_N^{(k)} = \frac{R}{A}[x + 2(\frac{A}{2R})^4 \tilde{S}_N^{(k)}],$$

which stands on the same footing as $S_R^{(k)}$ does in the $x_F >> 1$ limit. The density and the total mass in this iteration are given by the formulas (D.31) and (D.32) with $S_R^{(k)}$ replaced by $S_N^{(k)}$ of this limit, where

$$\dot{S}_N^{(k)} = \frac{R}{A}[1 + 2(\frac{A}{2R})^4 \dot{\tilde{S}}_N^{(k)}].$$

It is again obvious that this iteration is valid as long as $\frac{A}{2R} << 1$.

As we have seen in both cases, the unique feature of the present iterations is the fact that they both start from functions free of the parameter R and then introduce it as the iterations proceed.

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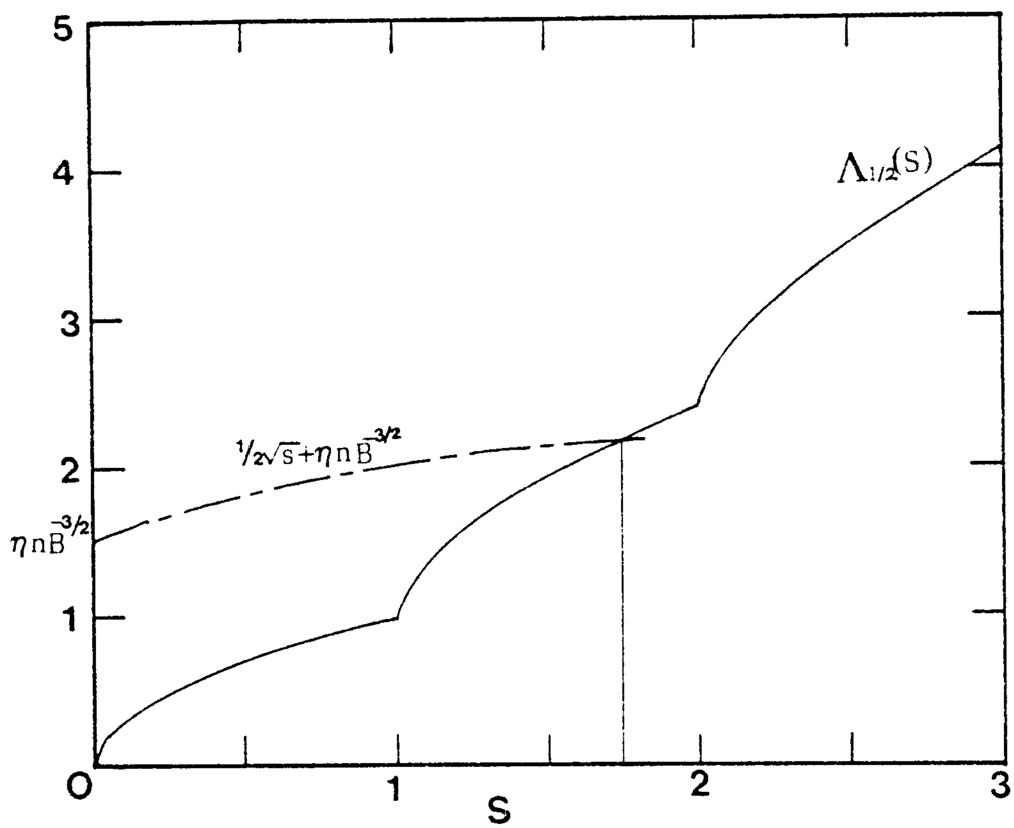


Fig. 1

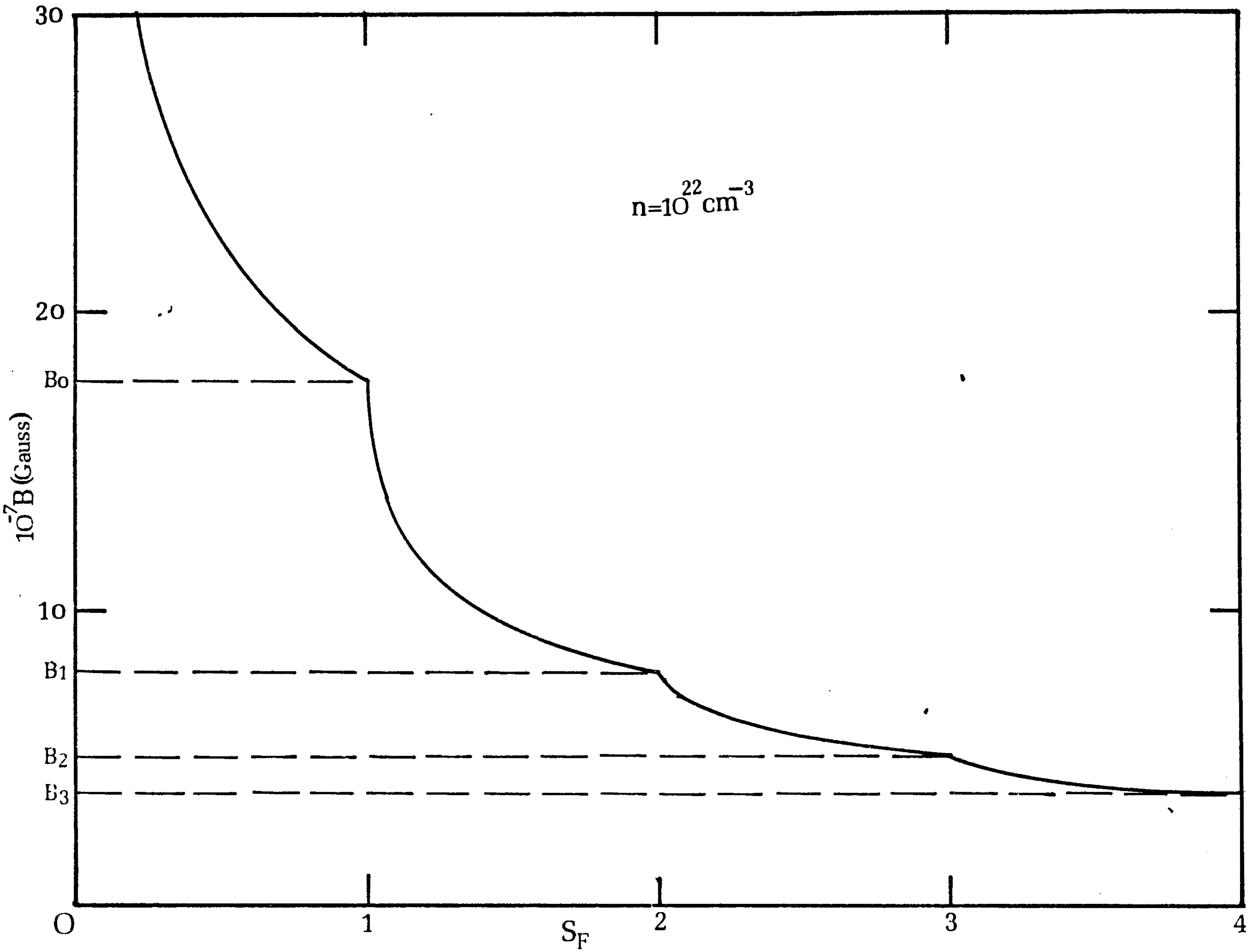


Fig. 2

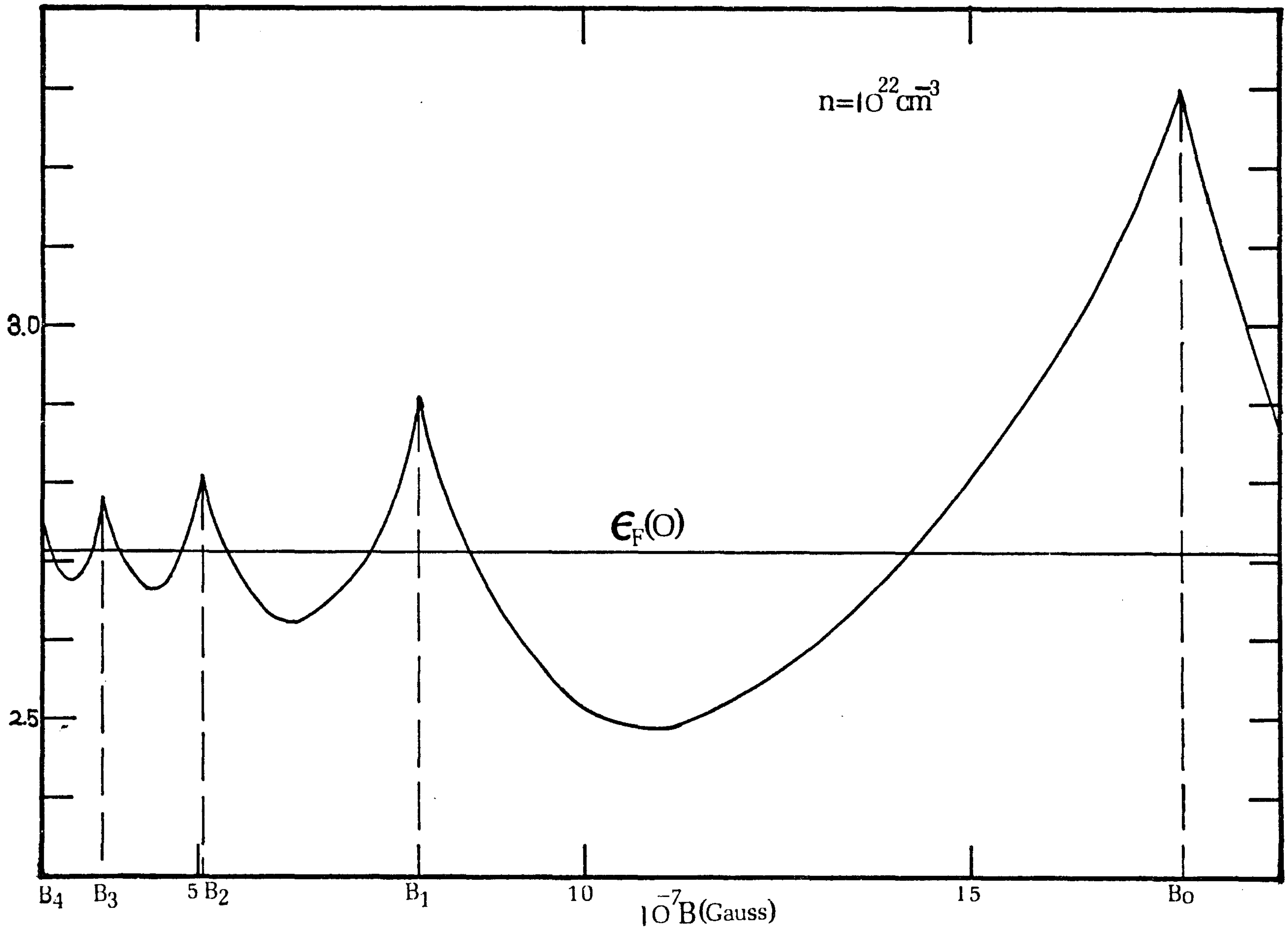
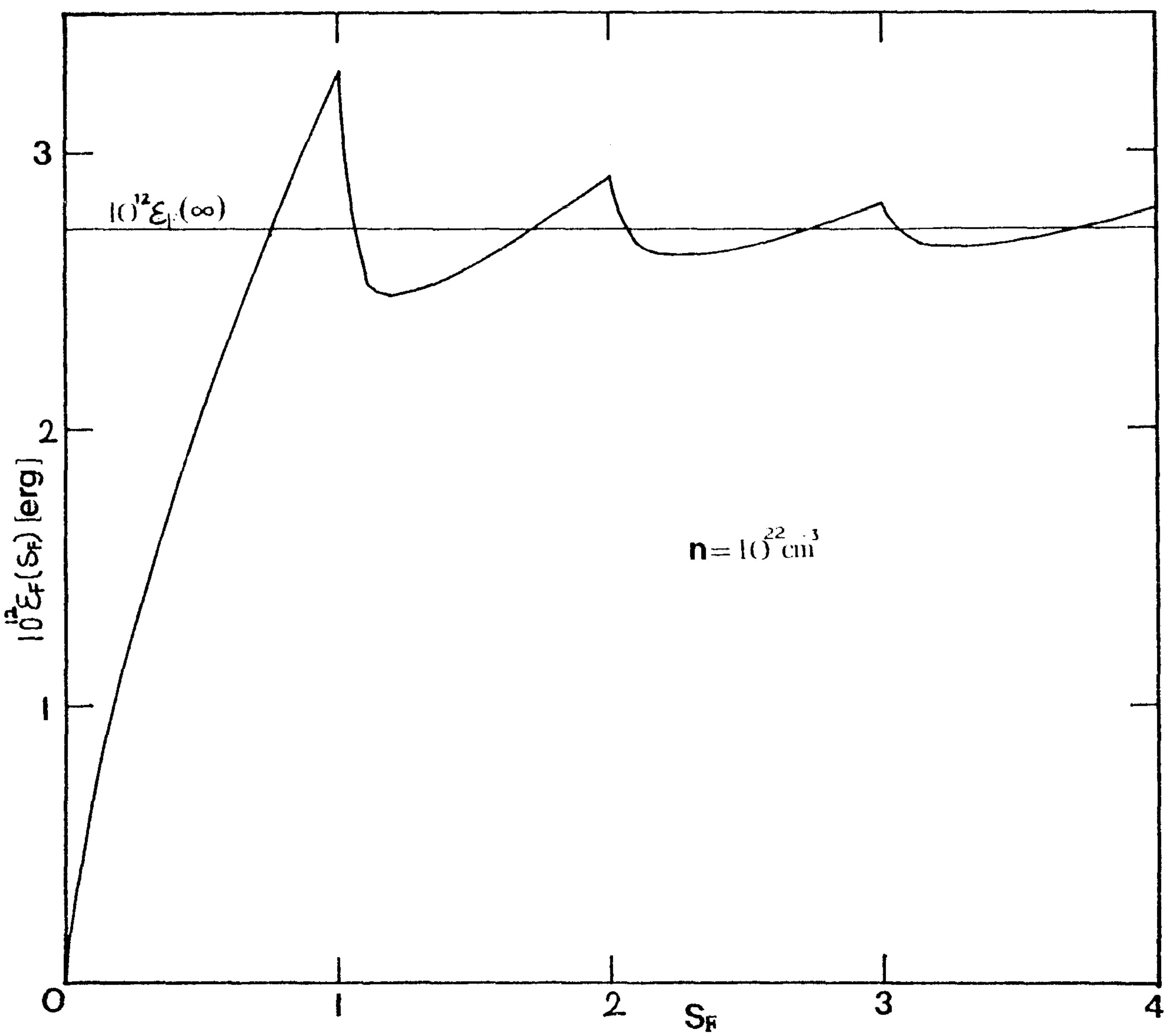


Fig. 3



3
2
1
0
1
2
3
4
S_F
$10^{12}\mathcal{E}_F(S_F)$ [erg]
$10^{12}\mathcal{E}_F(\infty)$
$n = 10^{22}\ cm^{-3}$

Fig. 4

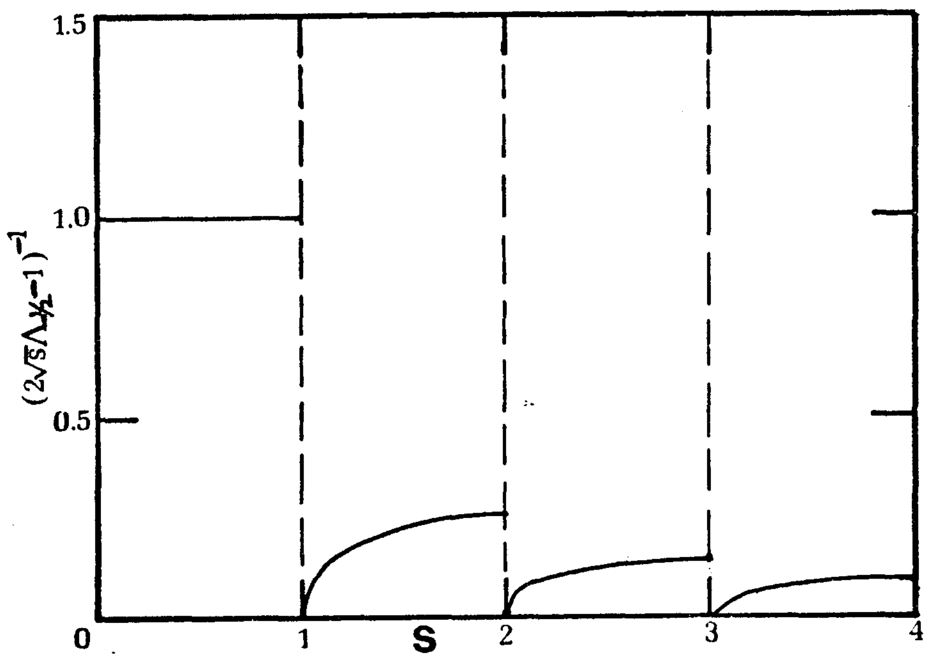


Fig. 5

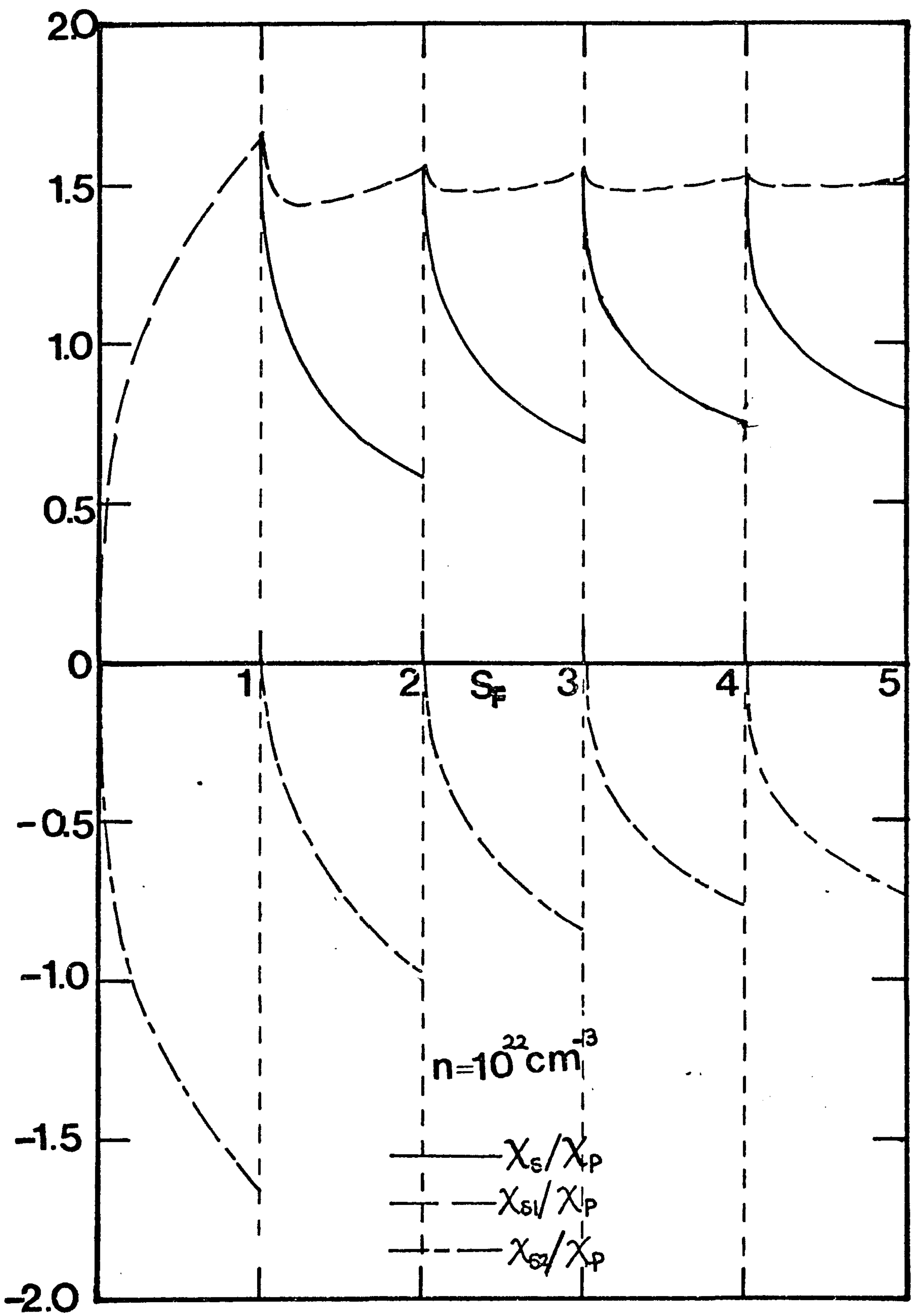

2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0
1
2
3
4
5
S_F
$n = 10^{22}\,cm^{-3}$
χ_s/χ_p
χ_{s1}/χ_p
χ_{s2}/χ_p

Fig. 6

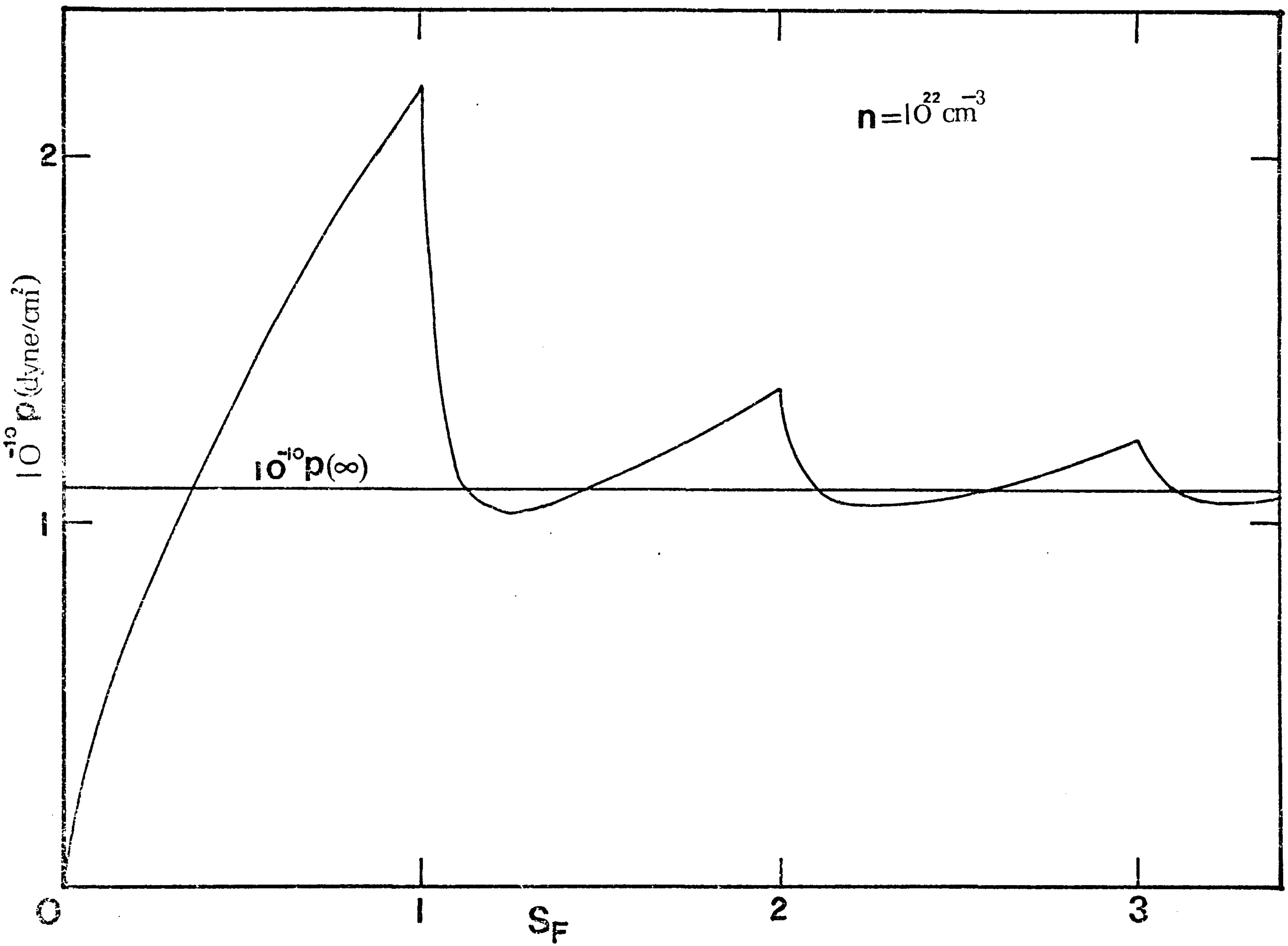


Fig. 7

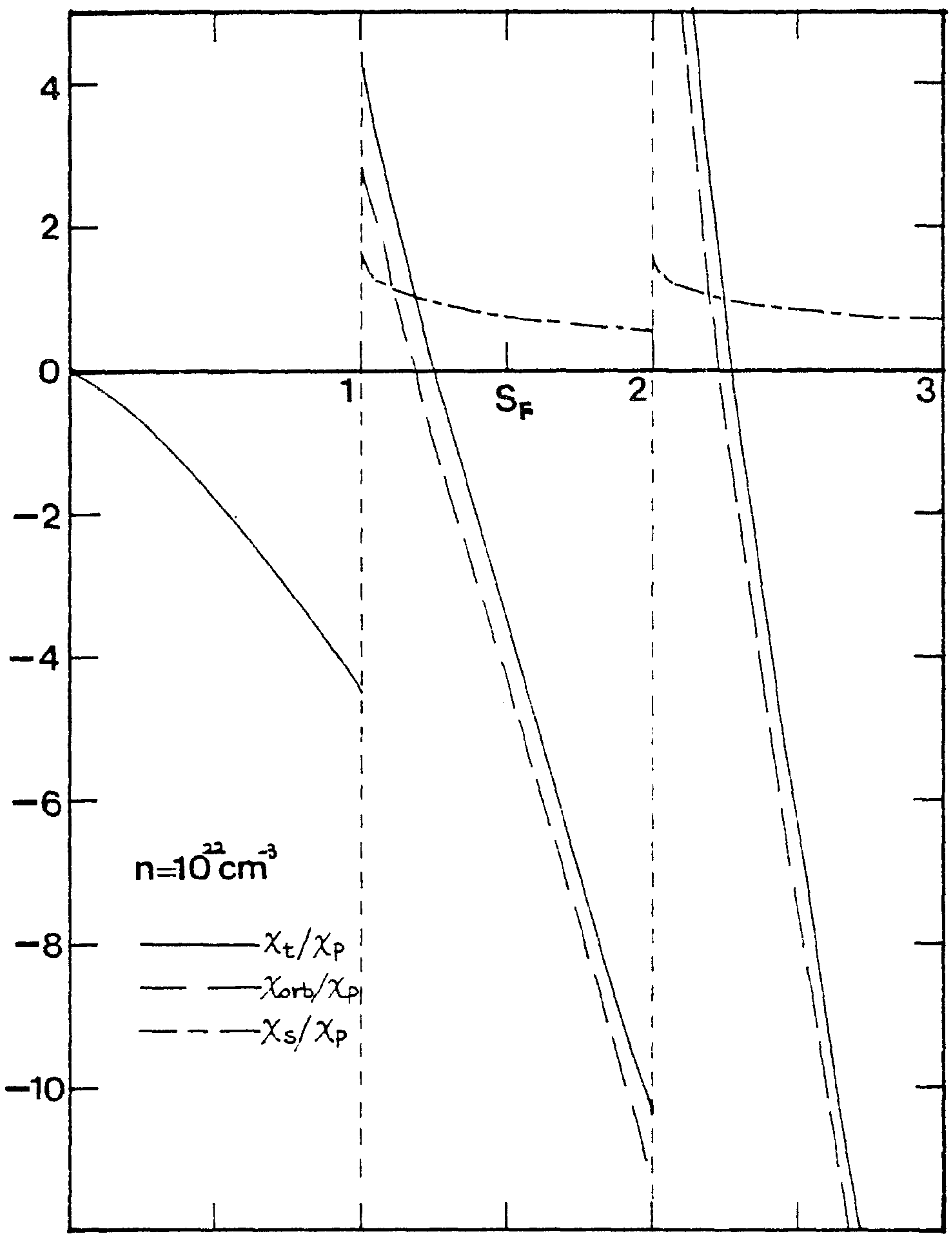


Fig. 8

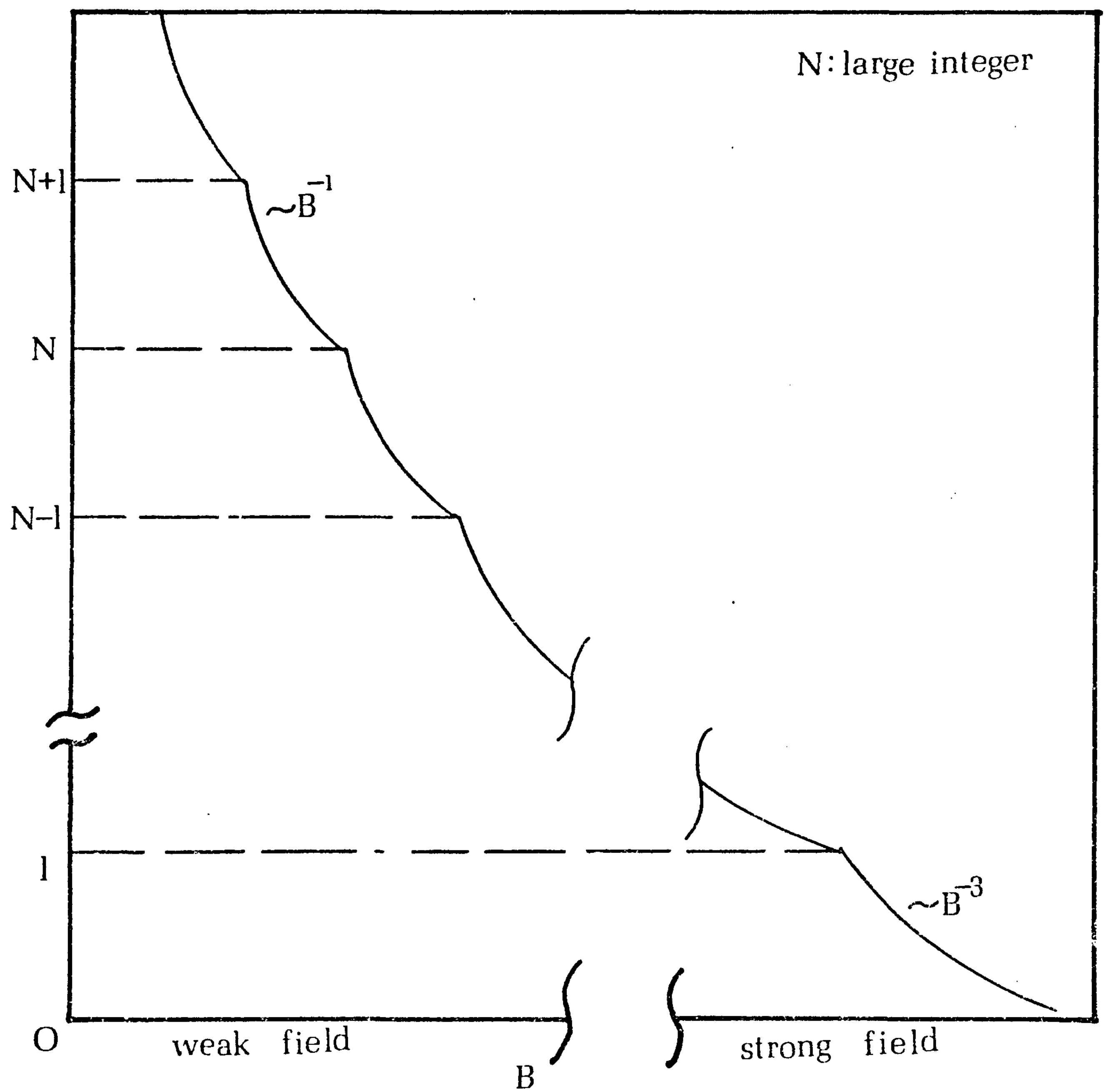


Fig. 9

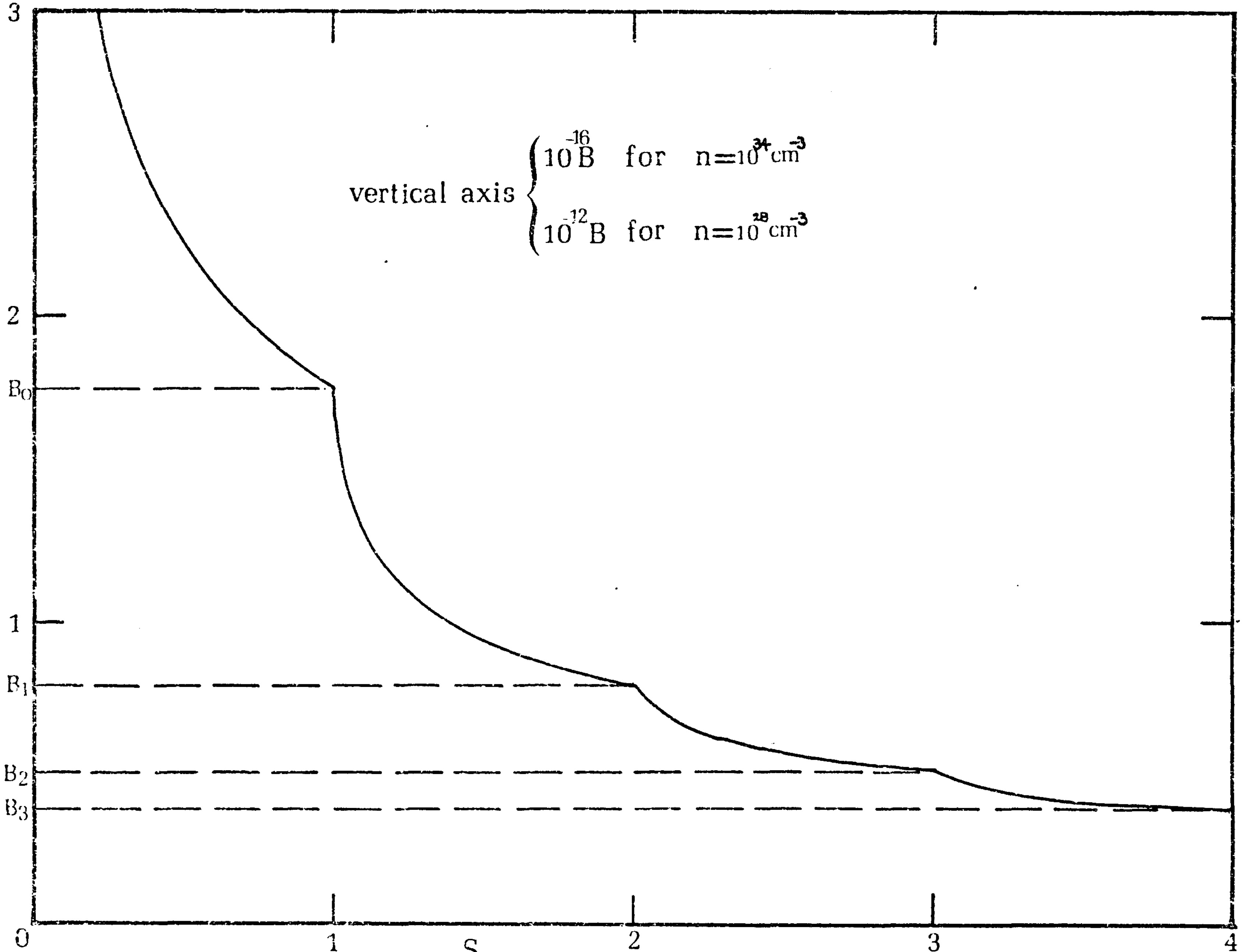

3
2
B_0
1
B_1
B_2
B_3
0
1
2
3
4
S_F
vertical axis
$10^{-16}B$ for $n=10^{34}cm^{-3}$
$10^{-12}B$ for $n=10^{28}cm^{-3}$

Fig. 10

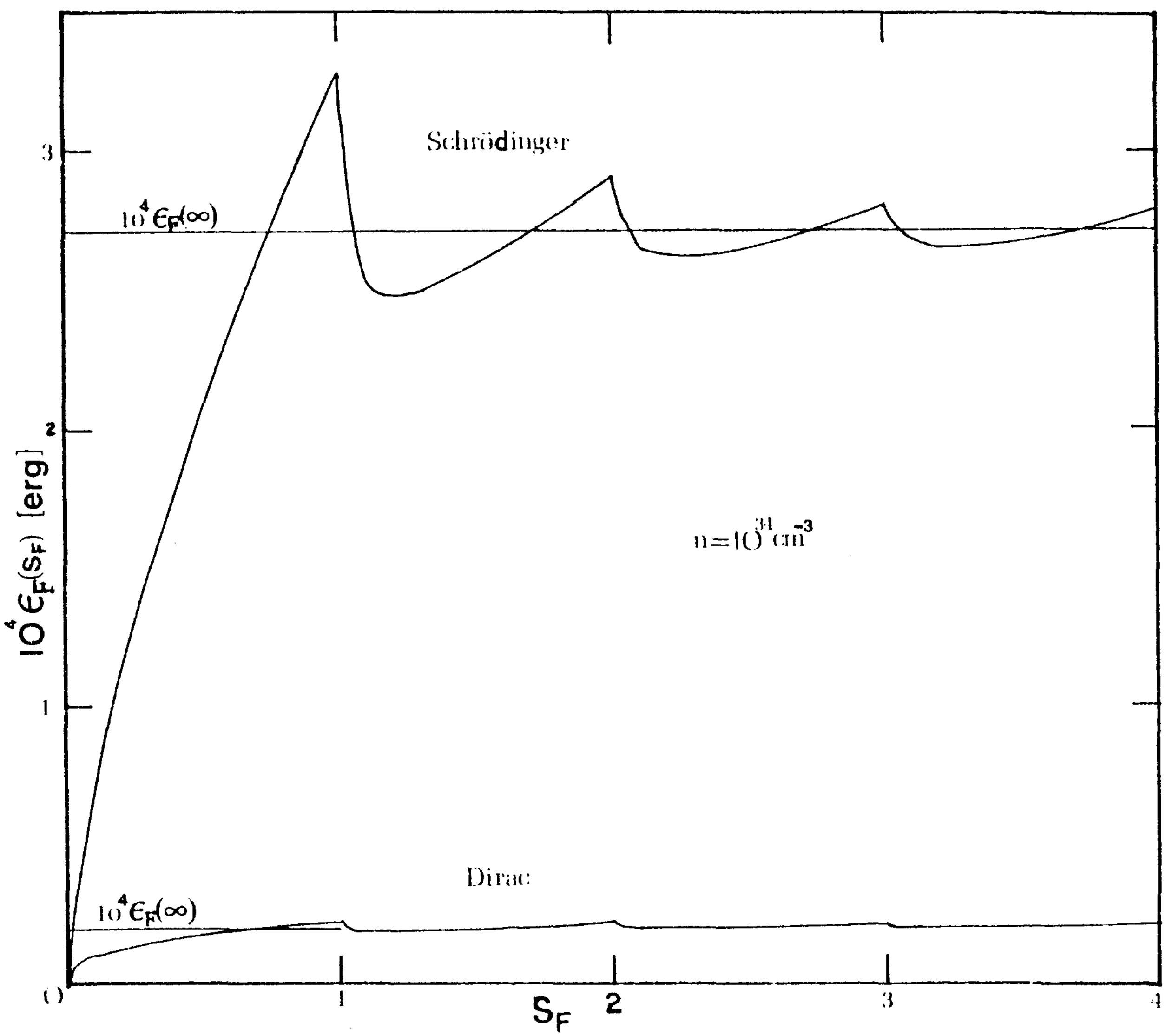

Schrödinger
$10^4\epsilon_F(\infty)$
$n=10^{34}\,cm^{-3}$
Dirac
$10^4\epsilon_F(\infty)$
$10^4\epsilon_F(s_F)$ [erg]
S_F
0
1
2
3
4
0
1
2
3

Fig. 11

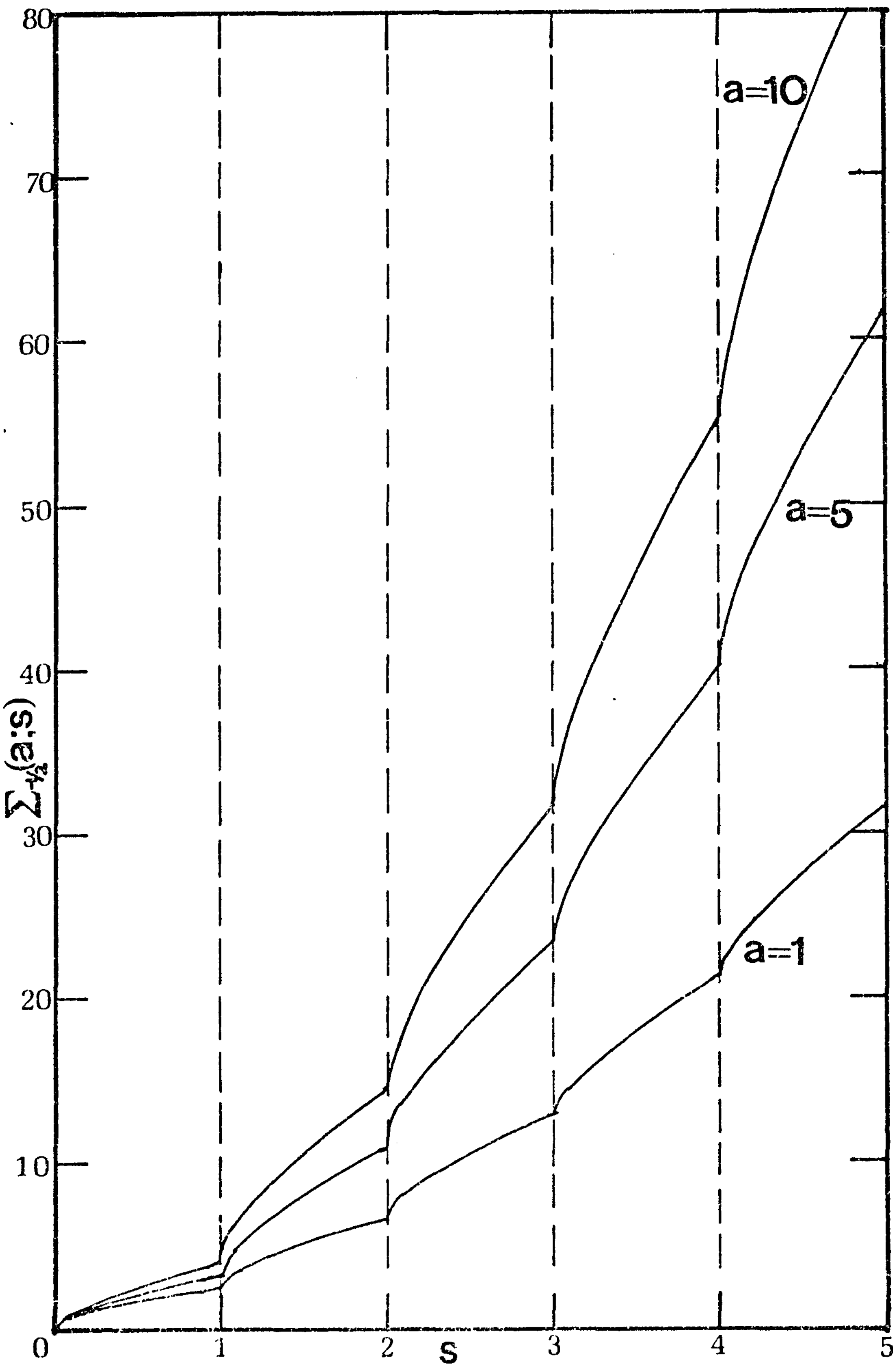


Fig. 12

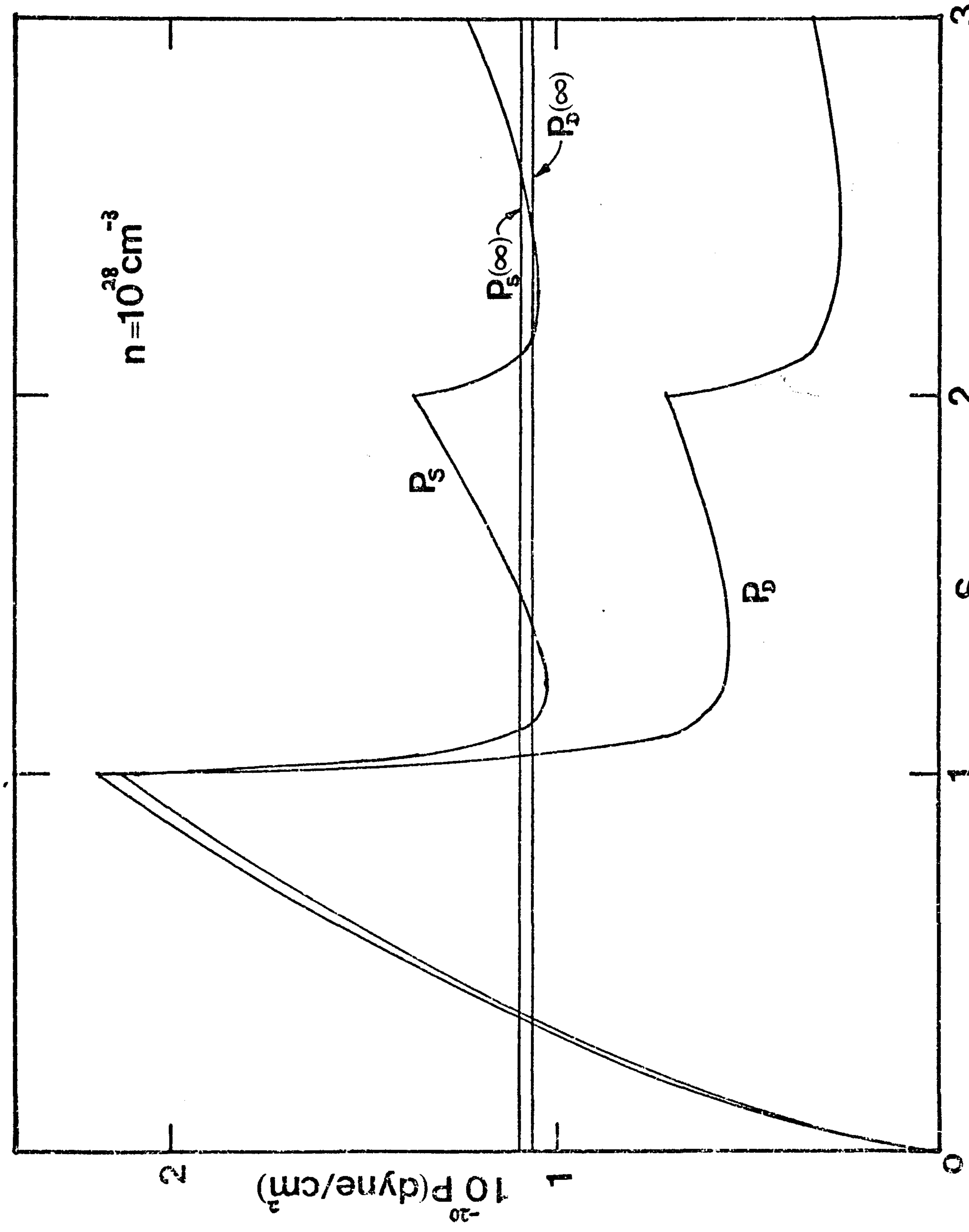


Fig. 13

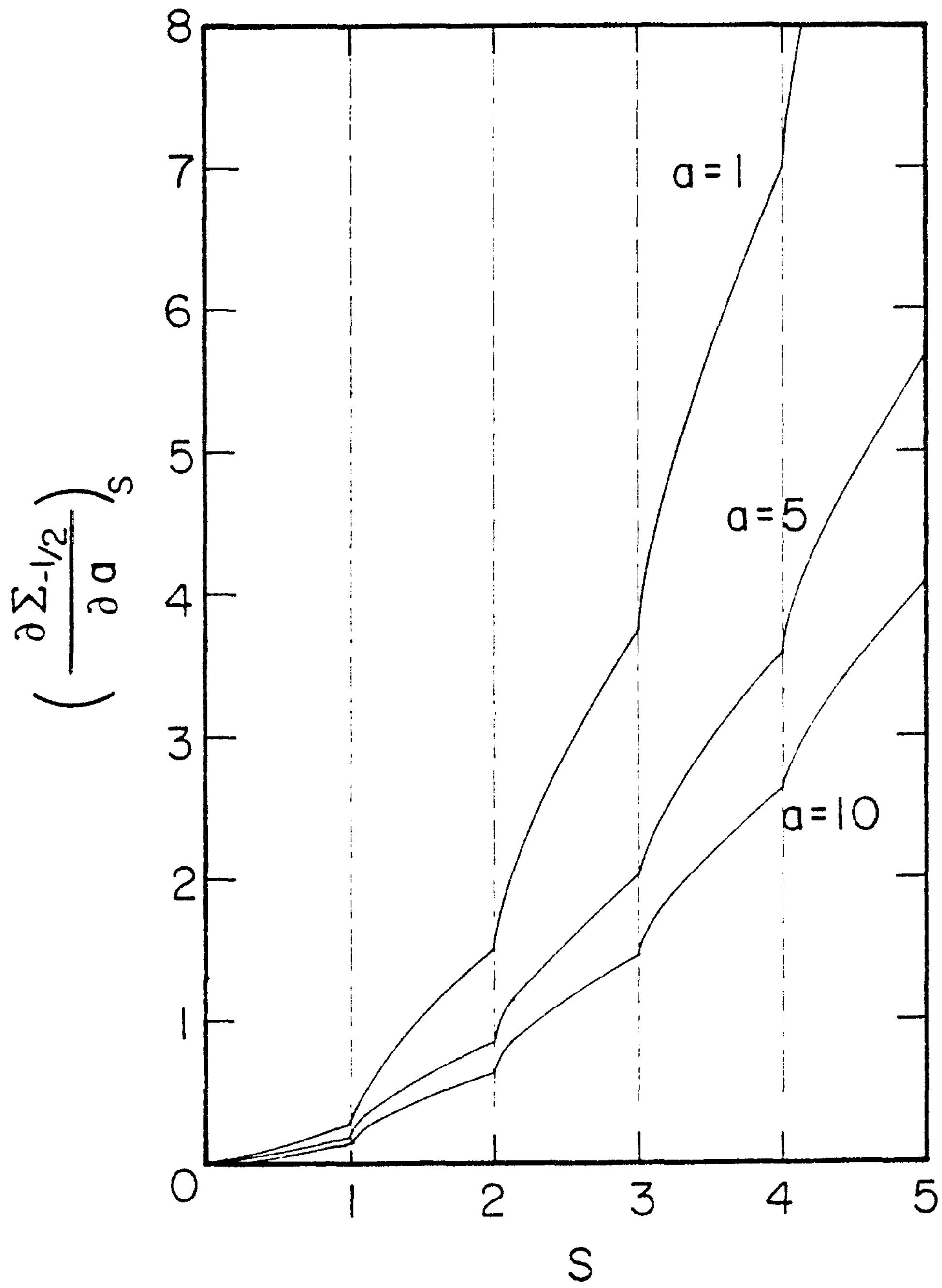

8
7
6
5
4
3
2
1
0
1
2
3
4
5
S
$\left(\frac{\partial \Sigma_{-1/2}}{\partial a}\right)_S$
a = 1
a = 5
a = 10

Fig. 14

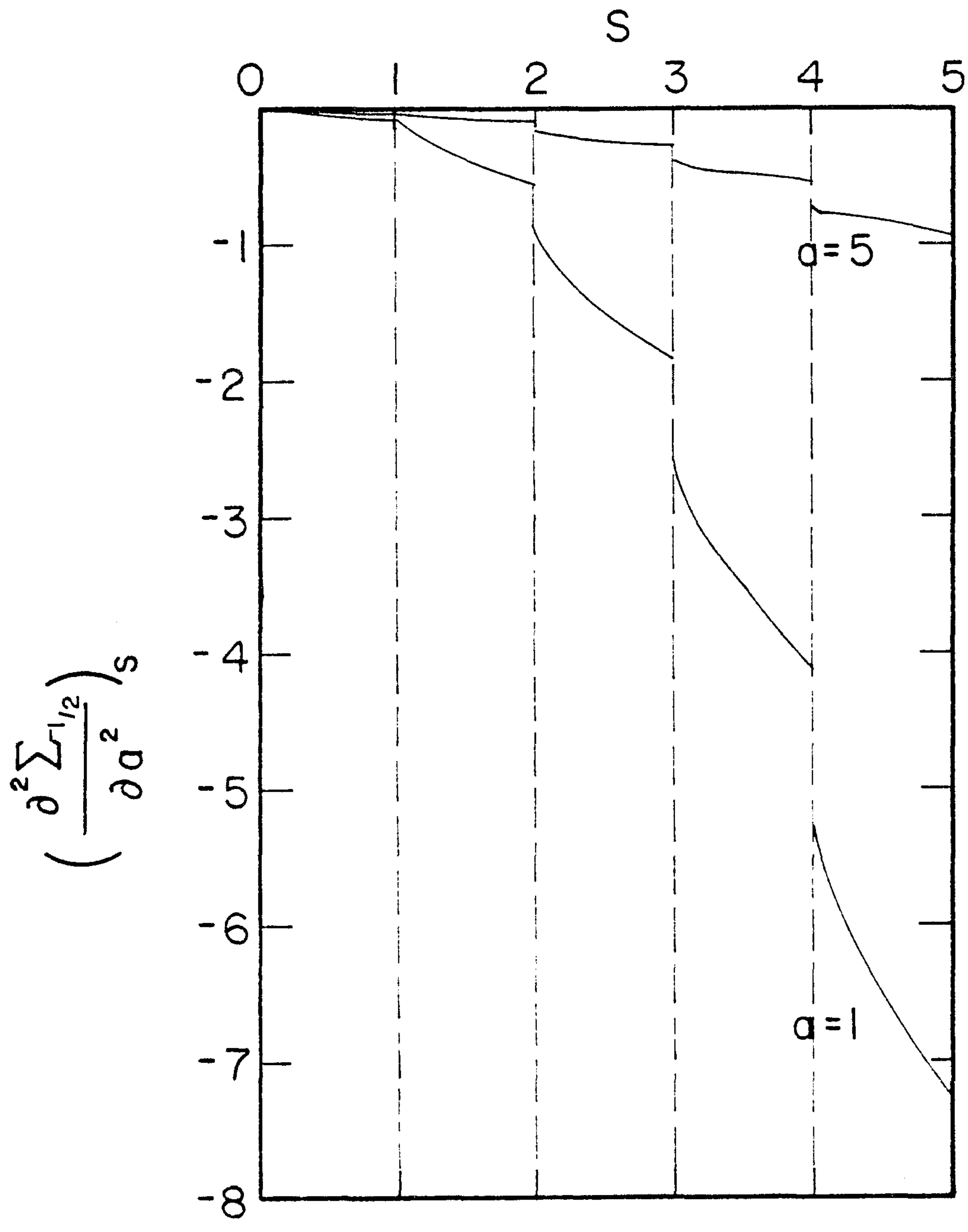
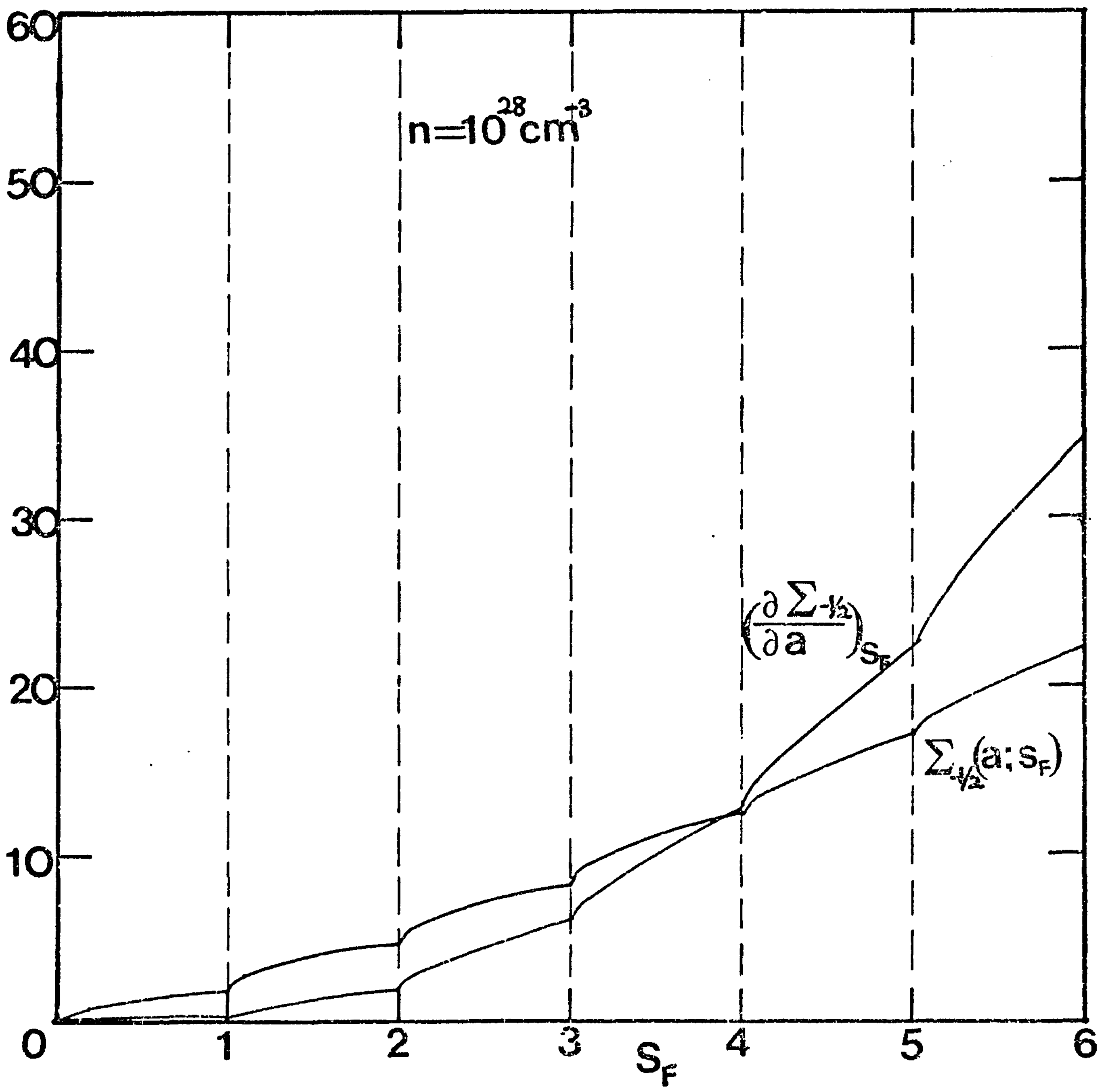

S
0
1
2
3
4
5
-1
-2
-3
-4
-5
-6
-7
-8
a=5
a=1
$\left(\frac{\partial^2 \Sigma_{-1/2}}{\partial a^2}\right)_s$

Fig. 15



60
50
40
30
20
10
0
1
2
3
4
5
6
$n = 10^{28}\,cm^{-3}$
$\left(\frac{\partial \Sigma_{-1/2}}{\partial a}\right)_{s_F}$
$\Sigma_{-1/2}(a; s_F)$
s_F

Fig. 16

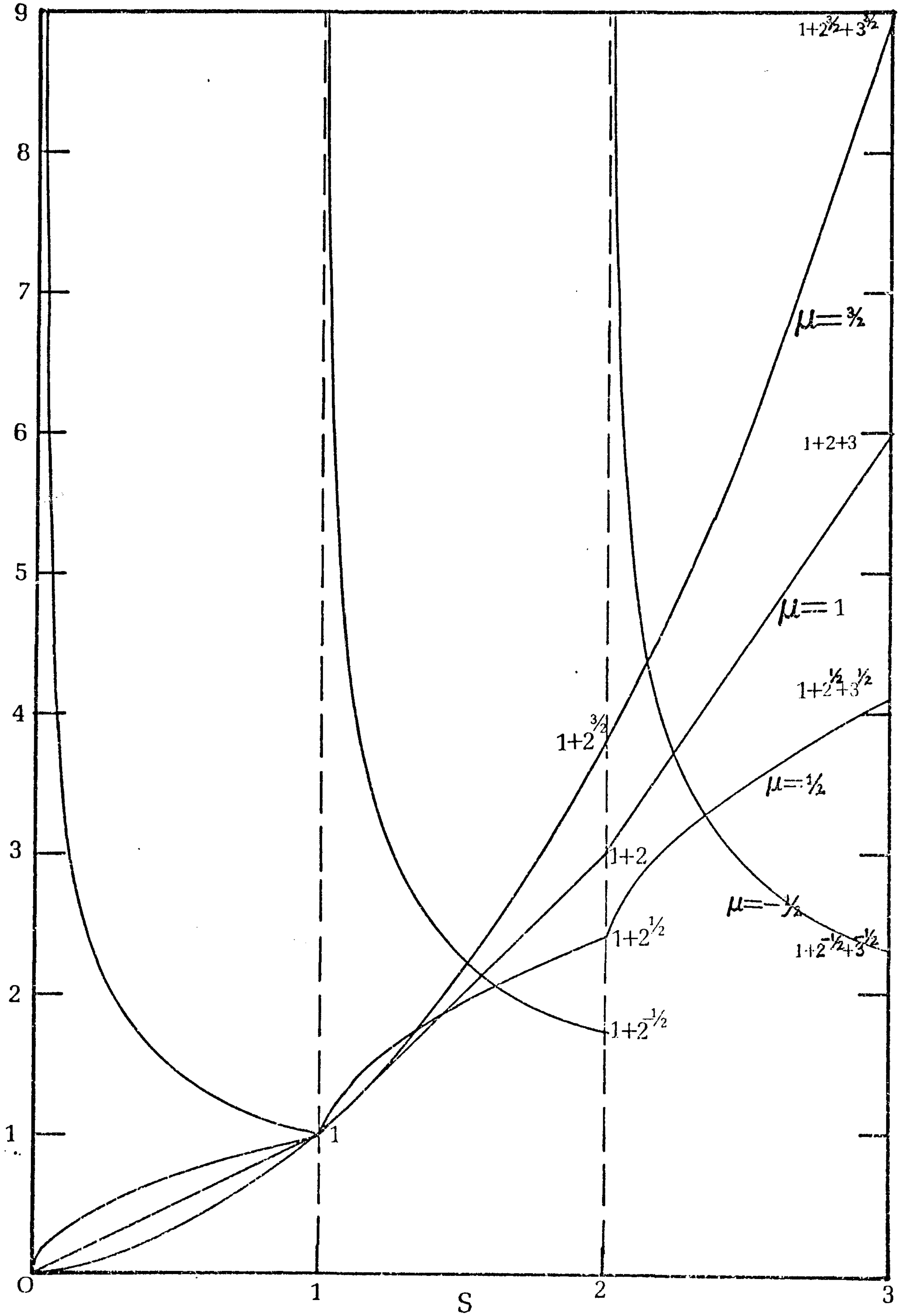

9
8
7
6
5
4
3
2
1
O
1
2
3
S
$1+2^{3/2}+3^{3/2}$
$\mu = 3/2$
$1+2+3$
$\mu = 1$
$1+2^{1/2}+3^{1/2}$
$1+2^{3/2}$
$\mu = 1/2$
$1+2$
$\mu = -1/2$
$1+2^{1/2}$
$1+2^{-1/2}+3^{-1/2}$
$1+2^{-1/2}$
1

Fig. 17

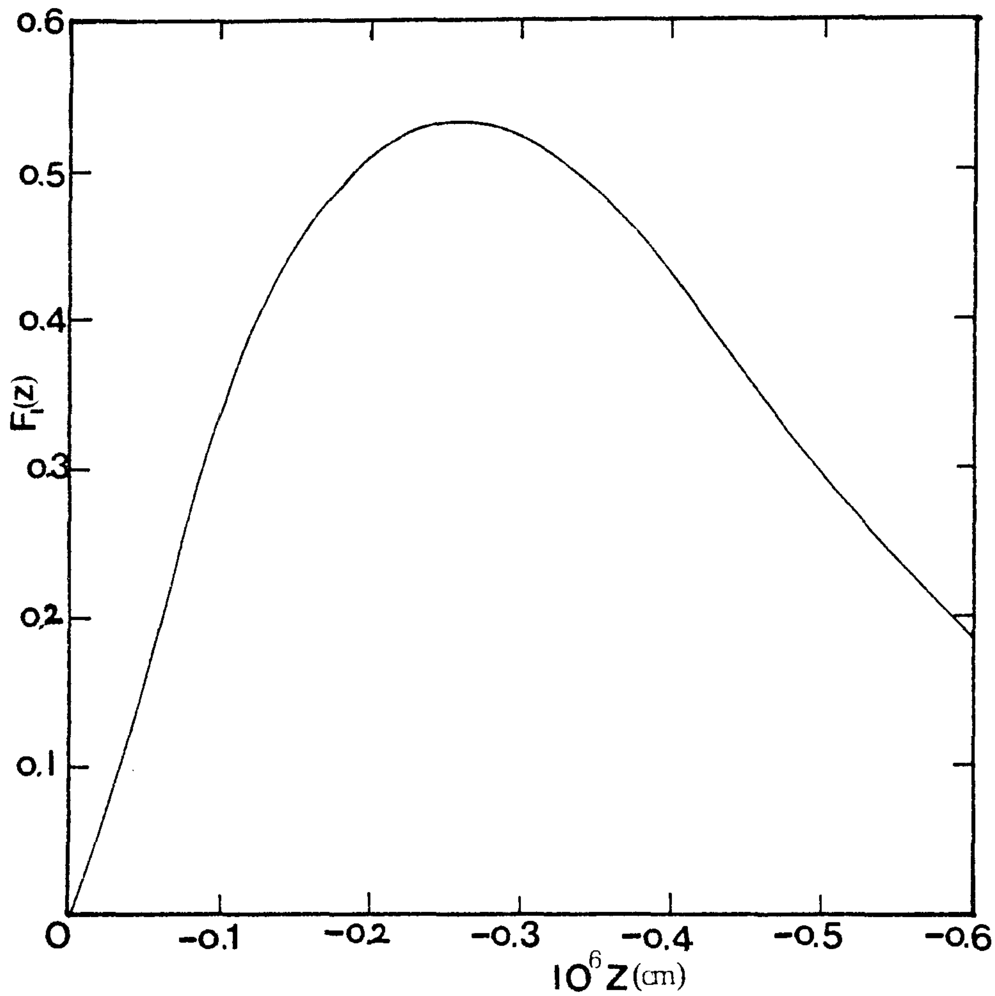


Fig. 18

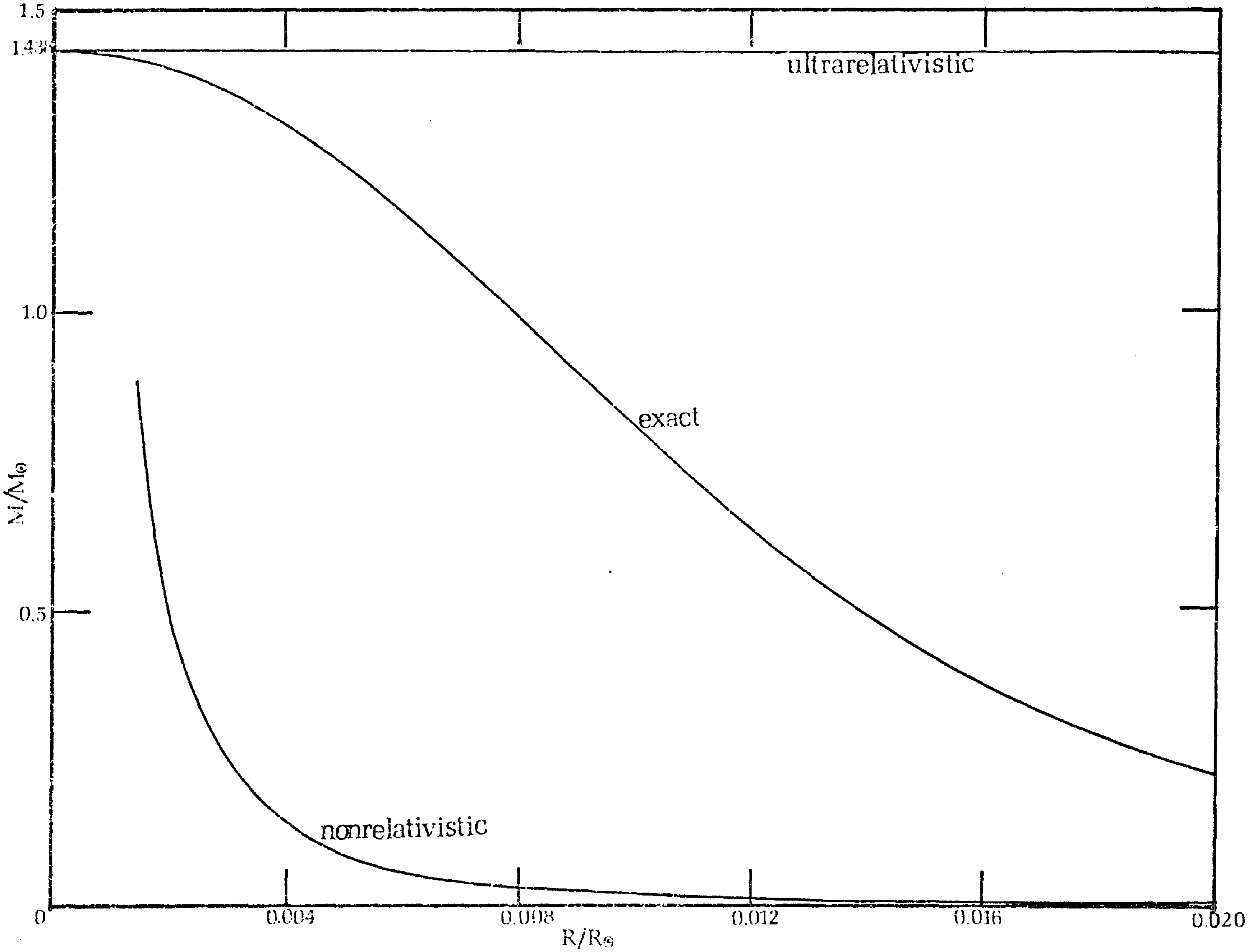

1.5
1.0
0.5
0
0.004
0.008
0.012
0.016
0.020
R/R⊙
M/M⊙
ultrarelativistic
exact
nonrelativistic

Fig. 19

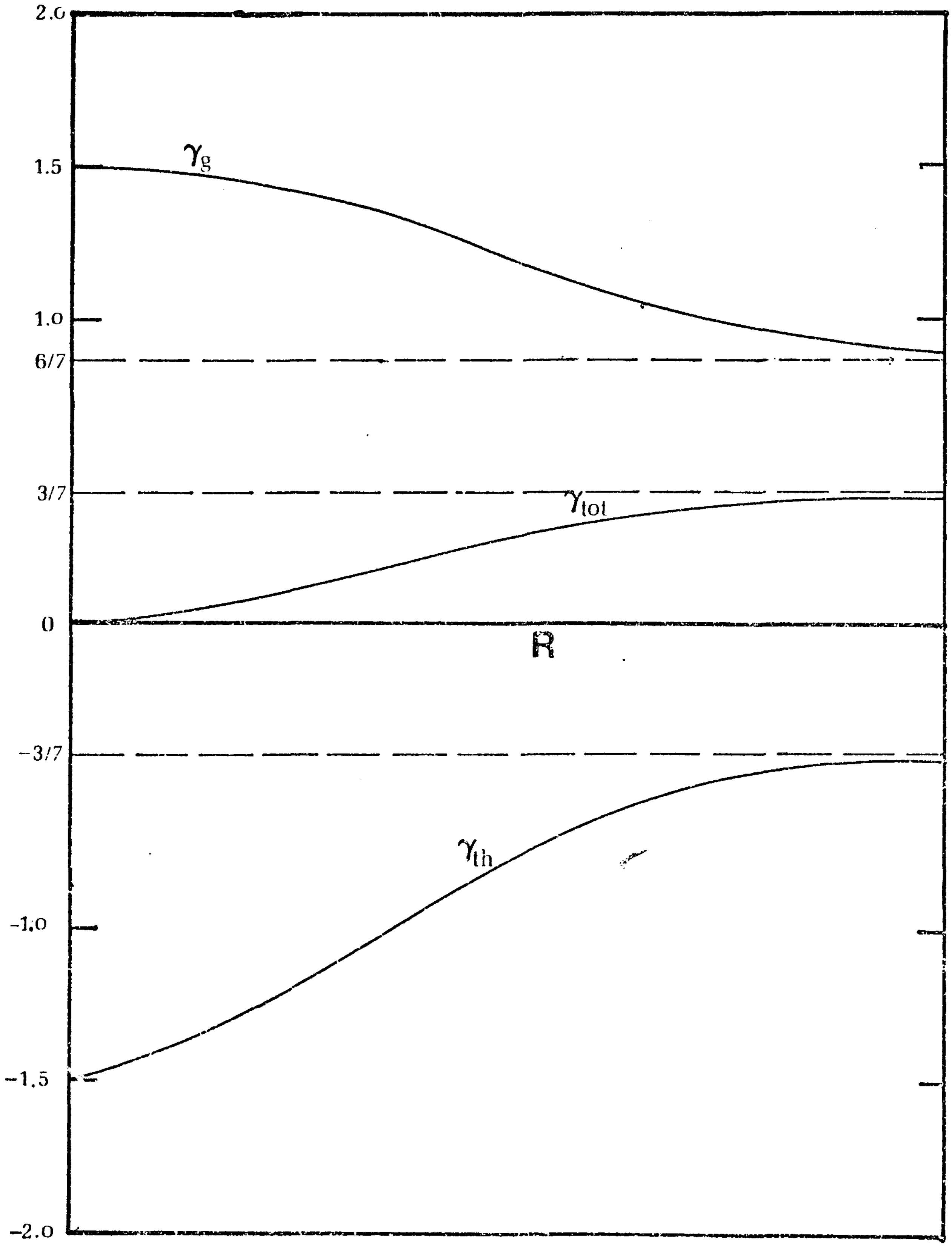


Fig. 20

PART II

THE PATH OF A SOUND WAVE IN GENERAL RELATIVITY

CHAPTER I

INTRODUCTION

It has been well established that a path of light ray deflects towards stronger gravitational fields.[1,2] However, nothing is known in general relativity about the path of sound waves in the gravitational field. The acoustic wave is as good a signal as a light ray in case of material media. Here we investigate the path of a sound wave in the general theory of relativity (GTR).

As a counterpart of geometrical optics there is geometrical acoustics.[3] However we have to realize that unlike the case of a light wave the sound wave requires a material medium of density $\rho(\underset{\sim}{r})$. We note that for nonuniform speed of sound $v(\underset{\sim}{r})$ Newtonian mechanics, or the special theory of relativity (STR), predicts the deflection of a sound path resulting from the famous Fermat principle regardless of the existence of a gravitational field in the medium.

In GTR the path is deflected due to the nature of curved space even for a medium of uniform $v(\underset{\sim}{r})$; i.e., the sound wave follows the geodesic[4] of Riemannian space. This geodesic is of course different from that of Euclidean space. We first consider this special case of uniform speed of sound, $v(\underset{\sim}{r})$ = constant, throughout the medium. The sound path for this case in STR describes a straight line, the geodesic in Euclidean space. The path should be a geodesic in Riemannian space in GTR when the gravitational field is turned on. The path satisfies

$$\frac{d^2\chi^i}{ds^2} + \begin{Bmatrix} i \\ j\ k \end{Bmatrix} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad ds^2 = g_{ik}dx^i dx^k, \qquad i,j,k = 0,1,2,3. \tag{1.1}$$

Notice that s is a good parameter for a sound wave since $v(\underset{\sim}{r}) < c$ and $ds^2 \neq 0$. This Eq. (1.1) is equivalent to the variational integral

$$\delta\int ds = 0 \text{ or } \delta\int g_{ik}\dot{x}^i\dot{x}^k dx = 0, \quad \dot{x}^i = \frac{dx^i}{ds}\ . \tag{1.2}$$

Alternatively we can derive Eq. (1.1) directly from the wavevector k^i. Since $dk^i = 0$ in STR we obtain $Dk^i = 0$ in GTR.[6] Introducing the parameter λ such that $k^i = \frac{dx^i}{d\lambda}$, the equation $Dk^i = 0$ yields

$$\frac{dk^i}{d\lambda} + \begin{Bmatrix} i \\ j\ k \end{Bmatrix} k^j k^i = 0. \tag{1.3}$$

The parameter λ is proportional to the length parameter of the trajectory. Eqs. (1.1,2,3) are equivalent and describe the path of a sound wave in the medium of uniform speed of sound.

In general situations, where the speed $v(\underset{\sim}{r})$ is not uniform, we should employ the Fermat principle which we now derive. Here we assume the medium produces a constant gravitational field which is static or stationary field at worst. Defining the contra-vector $k^i = (\frac{\omega}{v}, \underset{\sim}{k})$, where

$|\underset{\sim}{k}| = \frac{\omega}{v}$, we recall the Fermat principle[7] in STR which reads

$$\delta \int k_\alpha dx^\alpha = 0, \quad \alpha = 1,2,3. \tag{1.4}$$

The time component of the covector k_o becomes

$$k_o = \frac{\omega_o}{v} = g_{oi}k^i = g_{oo}k^o + g_{o\alpha}k^\alpha = g_{oo}(k^o + \frac{g_{o\alpha}}{g_{oo}}k^\alpha)$$

$$= g_{oo}(k^o - g_\alpha k^\alpha),$$

where $g_\alpha = -\frac{g_{o\alpha}}{g_{oo}}$ and ω_o is some characteristic angular frequency invariant along the path. Now $k^i k_i = 0$ becomes $g_{oo}(k^o - g_\alpha k^\alpha)^2 - \gamma_{\alpha\beta}k^\alpha k^\beta = 0$ or $\frac{1}{g_{oo}}(\frac{\omega_o}{v})^2 - \gamma_{\alpha\beta}k^\alpha k^\beta = 0$, where $\gamma_{\alpha\beta} = -(g_{\alpha\beta} - \frac{g_{o\alpha}g_{o\beta}}{g_{oo}})$. This yields $k^\alpha = \frac{\omega_o}{v}\frac{1}{\sqrt{g_{oo}}}\frac{dx^\alpha}{d\ell}$, where $d\ell^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta$ is the spatial line element, so that

$$k_\alpha = g_{\alpha i}k^i = g_{\alpha o}k^o + g_{\alpha\beta}k^\beta = -\frac{\omega_o}{v}(\frac{\gamma_{\alpha\beta}}{\sqrt{g_{oo}}}\frac{dx^\beta}{d\ell} + g_\alpha). \tag{1.5}$$

Combining Eq. (1.4) with Eq. (1.5) yields

$$\delta \int \frac{1}{v}(\frac{d\ell}{\sqrt{g_{oo}}} + g_\alpha dx^\alpha) = 0. \tag{1.6}$$

In the static field case ($g_\alpha = 0$) this reduces to

$$\delta \int \frac{d\ell}{v\sqrt{g_{oo}}} = 0. \tag{1.7}$$

Solving Eqs. (1.6) or (1.7) will provide the path equation in the stationary or static fields. Notice that Eqs. (1.6) and (1.7) are quite general so that one can use them for both an internal sound wave or an external light ray with v replaced by c in the latter case.[8]

CHAPTER II

THE PATH EQUATION

We now focus on the sound path for the spherically symmetric isotropic fluid. Then

$$ds^2 = e^{\nu(r)}c^2dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2,$$

where the spatial metric $\gamma_{\alpha\beta}$ reads

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} = \begin{pmatrix} e^{\lambda(r)} & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

for this static field. Equation (1.7) becomes

$$\delta\int\frac{1}{\sqrt{v^2g_{oo}}}\sqrt{\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\,dq = 0,$$

where $\dot{x}^\alpha = \frac{dx^\alpha}{dq}$ and q is an arbitrary non-vanishing parameter. Therefore we arrive at Euler-Lagrange differential equations

$$\frac{d}{dq}\left(\frac{\partial L}{\partial\dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha},\ \alpha = 1,2,3, \tag{2.1}$$

where $L = \sqrt{\frac{\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}{v^2g_{oo}}}$. We first calculate

$$\begin{aligned}\frac{d}{dq}\ell^2 = \frac{d}{dq}(\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) &= \dot{r}(\dot{r}^2\frac{de^\lambda}{dr} + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2) \\ &+ \dot{\theta}r^2\dot{\phi}^2\sin2\theta + 2\dot{r}\ddot{r}e^\lambda + 2\dot{\theta}\ddot{\theta}r^2 + 2r^2\sin^2\theta\dot{\phi}\ddot{\phi}.\end{aligned} \tag{2.2}$$

(i) r-Equation

The left- and right-hand sides (LHS and RHS) of Eq.

(2.1) become

$$\text{LHS} = \frac{1}{\sqrt{v^2 g_{oo}}} \quad \frac{\sqrt{\dot{\ell}^2}\,(\ddot{r}e^{\lambda} + \dot{r}^2 \frac{de^{\lambda}}{dr}) - \dot{r}e^{\lambda}\frac{1}{2\sqrt{\dot{\ell}^2}}\frac{d\dot{\ell}^2}{dq}}{2}$$

$$- \frac{\dot{r}^2 e^{\lambda} \frac{d}{dr}(v^2 g_{oo})}{2(\dot{\ell}^2 v^2 g_{oo})^{1/2}\ v^2 g_{oo}}$$

$$\text{RHS} = -\sqrt{\dot{\ell}^2}\ \frac{\frac{d}{dr}(v^2 g_{oo})}{2(v^2 g_{oo})^{3/2}} + \frac{1}{2(v^2\dot{\ell}^2 g_{oo})^{1/2}} \times \qquad (2.3)$$

$$\times\ (\dot{r}^2 \frac{de^{\lambda}}{dr} + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2)$$

(ii) θ-Equation

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} - \frac{r^2\dot{\theta}}{2\dot{\ell}^2}\ (\frac{d\dot{\ell}^2}{dq}) - \frac{r^2\dot{\theta}\dot{r}}{2v^2 g_{oo}}\ \frac{d}{dr}(v^2 g_{oo})$$
$$= r^2\sin\theta\cos\theta\dot{\phi}^2. \qquad (2.4)$$

(iii) φ-Equation

$$\frac{d}{dq}\ \frac{r^2\sin^2\theta\dot{\phi}}{(v^2\dot{\ell}^2 g_{oo})^{1/2}} = 0 \qquad (2.5)$$

or, expanding this, we get

$$2r\sin^2\theta\dot{\phi}\dot{\gamma} + r^2\sin^2\theta\ddot{\phi} + r^2\sin 2\theta\dot{\theta}\dot{\phi} - \frac{r^2\sin^2\theta\dot{\phi}}{2\dot{\ell}^2}\frac{d\dot{\ell}^2}{dq}$$
$$= \frac{r^2\sin^2\theta\dot{\phi}\dot{\gamma}}{2v^2 g_{oo}}\ \frac{d}{dr}(v^2 g_{oo}). \qquad (2.5')$$

Now we set $\theta = \pi/2$, $\dot{\theta} = 0$ and consider the path in the x-y plane for simplicity. Then Eq. (2.2) becomes

$$\frac{d\dot{\ell}^2}{dq} = \dot{\gamma}(\dot{\gamma}^2 \frac{de^\lambda}{dr} + 2r\dot{\phi}^2) + 2e^\lambda\dot{\gamma}\ddot{\gamma} + 2r^2\dot{\phi}\ddot{\phi} , \tag{2.6}$$

since $\dot{\ell}^2 = e^\lambda\dot{\gamma}^2 + r^2\dot{\phi}^2$. Now we notice that the r- and φ-Equations become identical and we can choose to work with either of these two. We take Eq. (2.5) for convenience. Herewith we write

$$\frac{r^2\dot{\phi}}{(v^2 g_{oo}\dot{\ell}^2)^{1/2}} = C_1,$$

where C_1 is some positive constant. Using $\gamma' \equiv \frac{dr}{d\phi} = \dot{r}/\dot{\phi}$ this becomes

$$r^2(\frac{\gamma^2}{C_1^2 v^2 g_{oo}} - 1)e^{-\lambda} = (\gamma')^2. \tag{2.7}$$

The two constants, including C_1 in Eq. (2.7), are to be determined by the two initial conditions, namely, the starting point (r_1, ϕ_1) and the slope of the propagation direction at this point, $S = (\frac{dy}{dx})_1$. Now we notice, without loss of generality, we can always choose $\phi_1 = 0$. This yields

$$S = (\frac{dy}{dx})_1 = r_1/(r')_1 \quad , \text{ or } (\frac{dr}{d\phi})_1 = \frac{r_1}{S}. \tag{2.8}$$

Combining Eq. (2.8) with Eq. (2.7) yields

$$C_1 = \frac{r_1}{v(r_1)\left[g_{oo}(r_1)\left(\frac{e^{\lambda(r_1)}}{S^2} + 1\right)\right]^{1/2}}, \; r_1 \neq 0. \tag{2.9}$$

Finally, integration of Eq. (2.7) from r_1 to r yields the desired path equation

$$\phi = \frac{C_1 S}{|S|} \int_{r_1}^{r} \frac{e^{\lambda/2} v\sqrt{g_{oo}}}{r\sqrt{r^2 - C_1^2 v^2 g_{oo}}}\, dr, \tag{2.10}$$

where C_1 is given by formula (2.9). Next we infer the physical meaning of the constant C_1 as expressed in terms of the polar coordinates of the closest point on the trajectory to the center, (r_o, ϕ_o). Obviously $(\frac{dr}{d\phi})_o = 0$ and so Eq. (2.7) gives

$$C_1 = \frac{r_o}{v_o\sqrt{g_{oo}(r_o)}}, \quad v_o \equiv v(r_o). \tag{2.11}$$

Furthermore, combining Eq. (2.9) with Eq. (2.11) we obtain the equation determining r_o uniquely:

$$r_o = \frac{v_o}{v_1}\left[\frac{g_{oo}(r_o)}{g_{oo}(r_1)}\right]^{1/2} \frac{r_1}{\left(\frac{e^{\lambda_1}}{S^2} + 1\right)^{1/2}}, \tag{2.12}$$

where $v_1 \equiv v(r_1)$, etc. ϕ_o is found from the path Eq. (2.10) to be

$$\phi_o = \frac{C_1 S}{|S|} \int_{r_1}^{r_o} \frac{e^{\lambda/2} v(r)\sqrt{g_{oo}}}{r\sqrt{r^2 - C_1^2 v^2 g_{oo}}}\, dr, \tag{2.13}$$

where C_1 and r_o are given by Eqs. (2.9) and (2.12).

To check the correctness of the foregoing equations we take the simplest system: uniform speed and no gravity. These imply $g_{oo} = e^{\lambda} = 1$ and then Eq. (2.10) reads

$$\phi = -C_1 v \int_{r_1}^{r} \frac{dr}{r\sqrt{r^2 - C_1^2 v^2}} = \left[\cos^{-1} \frac{C_1 v}{r}\right]_{r}^{r_1}.$$

Taking $r_1 = a$ and $S = -1$, where a is the radius of the star, this becomes $\phi = \pi/4 - \cos^{-1}\frac{1}{\sqrt{2}x}$, where $x = r/a$, which is a straight line of slope -1, as expected. From Eqs. (2.12,13) we determine that $r_o = a/\sqrt{2}$ and $\phi_o = 45°$, as should be the case.

Next we show that the path deflects towards the region of stronger gravitational field even in the medium of uniform speed of sound by the following simple argument. Rewriting Eq. (2.10) as

$$\phi = \frac{Sr_1}{|S| v_1 \sqrt{\frac{e^{\lambda_1}}{S^2}+1}} \int_{r_1}^{r} \frac{e^{\lambda/2} v \sqrt{h/h_1}\, dr}{r\sqrt{r^2 - \frac{r_1^2 v^2 (h/h_1)}{v_1^2 (e^{\lambda_1}/S^2+1)}}} ,$$

where $h = g_{oo}$ and $\lambda_1 = \lambda(r_1)$, we set $v = v_1 =$ constant. In the absence of the gravitational field, namely $e^{\lambda/2} = h = h_1 = 1$, the path is the straight line starting from r_1 with the initial slope S. However, in the presence of the gravitational field, namely $e^{\lambda/2} > 1$, we approximate $h/h_1 \simeq 1$ in the above equation. Then we immediately conclude that $\phi_{GTR} > \phi_{STR}$, which indicates the deflection towards the stronger gravitational field in the spherical medium.

Now we calculate the path equation explicitly for a particular stellar model obtained by Tolman,[9] namely Tolman's 4th interior solution. This is given by

$$ds^2 = (D+Cr^2/a^2)c^2dt^2 - \frac{D+2Cr^2/a^2}{(1-Cr^2/a^2)(D+Cr^2/a^2)} \times \\ \times\ dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{2.14}$$

where $C = 1/2T$, $D = 1-1.5/T$, $T = a/r_g$, $r_g = \frac{2GM}{c^2}$ is the gravitational radius and a is the radius of the star. The pressure P and density ρ are given by

$$a^2\chi P = \frac{3(1-x^2)}{4T(T-1.5+x^2)}, \\ a^2\chi c^2\rho = \frac{6-4.5/T+x^2/2T}{2T-3+2x^2} - \frac{x^2(1-x^2/2T)}{(T-1.5+x^2)^2}, \tag{2.15}$$

where $\chi = 8\pi G/c^4$, $x = r/a$, and we see that $P/c^2\rho < 1/3$ everywhere as should be the case of any physical substance.[10] The speed of sound for a fluid sphere[5] given by

$$v(x) = \sqrt{\frac{dP}{d\rho}} = c\sqrt{\frac{(1-0.5/T)(1-1.5/T+x^2/T)}{5-10/T+x^2/T+15/4T^2-x^2/2T^2}}, \tag{2.16}$$

which is always less than c if $a > r_g$ or $T > 1$. The pressure, density and the ratio $P/c^2\rho$ are shown in Figs. 1-3. The speed of sound wave in this stellar interior is also shown in Fig. 20.

In order to find the path of a sound wave, we take $x_1 = 0.5$ and $S = +2$ for simplicity. Then the path Eq.(2.10) becomes

$$\phi(x) = C_1 \int_{0.5}^{x} \frac{\bar{V}(t)\sqrt{\frac{D+2Ct^2}{1-Ct^2}}dt}{t\sqrt{t^2-\bar{V}(t)^2 C_1^2 (D+Ct^2)}},$$

$$C_1 = \frac{0.5}{\bar{V}_1\sqrt{\frac{D+0.5c}{4(1-0.25c)}+D+0.25c}}, \tag{2.17}$$

where $\bar{V} = V(x)/c$ and $\bar{V}_1 = \bar{V}(0.5)$. Next, to see the effect due to the gravitational field only on this path, we consider the sound path for a wave of the same speed $V(x)$ but in the absence of the gravitational field in the medium, namely $g_{oo} = e^{\lambda} = 1$, flat space. For this special case Eq. (2.10) reduces to

$$\phi^{(0)}(x) = C_1^{(0)} \int_{0.5}^{x} \frac{\bar{V}(t)dt}{t\sqrt{t^2-C_1^{(0)2}\bar{V}^2(t)}},$$

$$C_1^{(0)} = \frac{0.5}{\bar{V}_1\sqrt{5/4}} . \tag{2.18}$$

These two paths are compared with each other for two different masses but the same radius a, namely T = 2,4. These are shown in Figs. 4,5. These clearly demonstrate that the path of sound wave deflects towards the stronger gravitational field in the medium and this effect becomes greater for the star of larger density.

There arises a question whether the system remains in thermodynamic equilibrium under the passage of the sound wave with a speed comparable to the speed of light c. Since the formula[5] $v = \sqrt{\frac{dP}{d\rho}}$ was derived under the assumption of a reversible adiabatic process during the propagation of the sound wave, we may obtain a relation between the relaxation time of the medium, τ, and the radius a. Although the adiabatic condition is satisfied for large speeds, the reversibility condition imposes the restriction $\tau << \frac{\lambda}{v} << \frac{a}{v}$, or $v\tau << a$, throughout the interior.

It should be emphasized that our derivation is quite general and not restricted to the interior starting point, but applies both to interior and exterior propagation. Therefore, for the light signal exterior to the star, we only have to replace v(r) by c and use proper metric tensor in all formulae.

Finally we study the effect of the polarization of the medium on the sound path using the Einstein-Cartan theory of gravitation.[11] We use the relations proved by Suh,[12] namely, $g_{oo}]_{EC} = g_{oo}]_{ES}$ and $-g_{rr}]_{EC} < -g_{rr}]_{ES}$ for the same set of $P(\rho)$, $\rho(r)$, and a. EC and ES stand for Einstein-Cartan and Einstein theory of gravitation respectively. Here we consider two interiors of the same set of $P(\rho)$, $\rho(r)$, and a, which give the same values of v(r), the speed of sound wave.

Since r_o in Eq. (2.11) can always be set equal to r_1 by some particular choice of S, the initial slope of the path at r_1, $C_1 = \frac{r_1}{v_1\sqrt{g_{oo}(r_1)}}$ has the same value for both theories. Since the only different quantity in the path Eq. (2.10) is $e^{\lambda/2}$ in the integrand, we conclude that $\phi_{ES}(r) > \phi_{EC}(r)$ by means of the above inequality. Physically this means that the path will deflect less in the polarized medium than in the medium without polarization.

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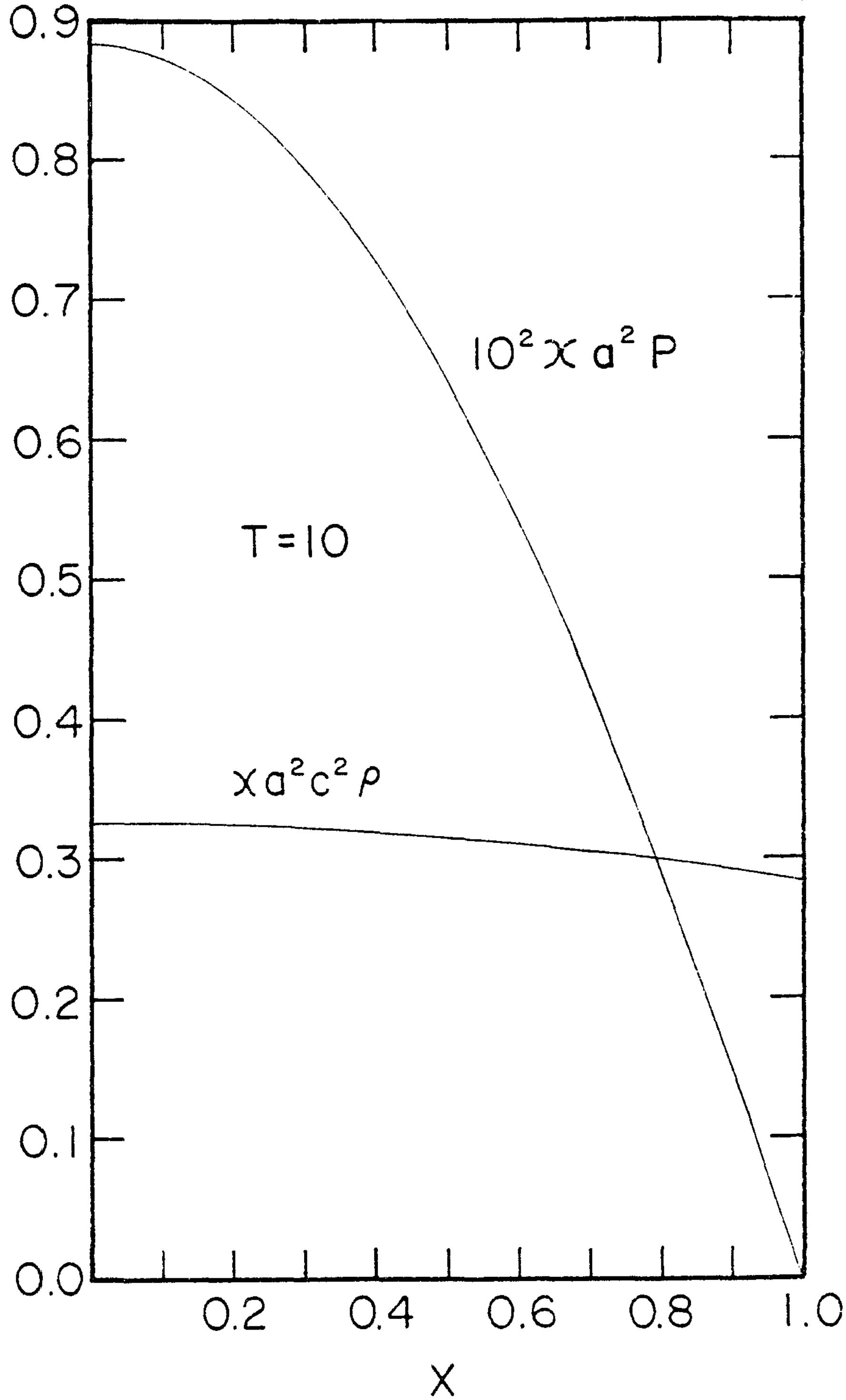
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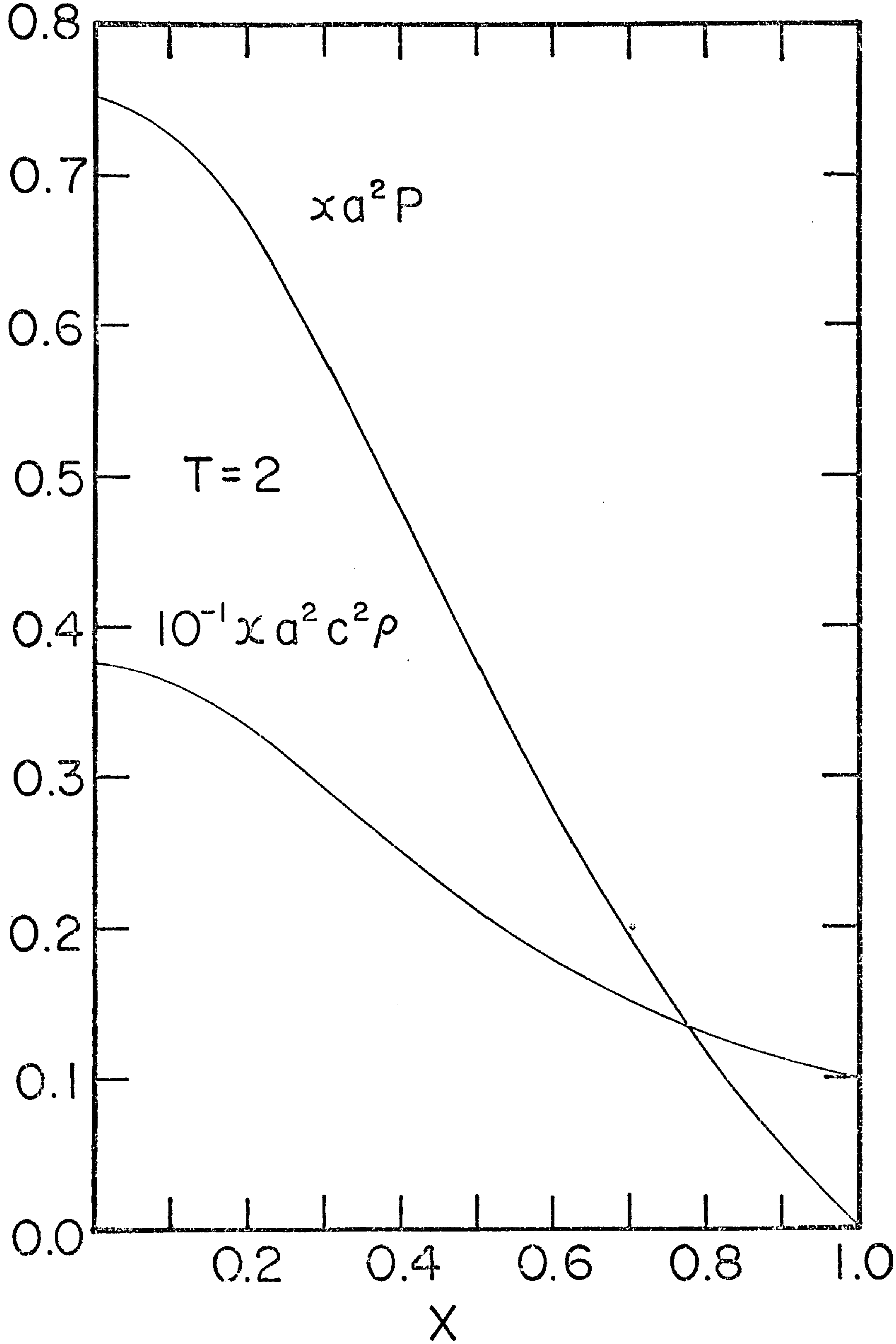
1.0
X
$10^2 \times a^2 P$
T=10
$\chi a^2 c^2 \rho$

Fig. 1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0.2
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0.8
1.0
X
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T = 2
$10^{-1} \chi a^2 c^2 \rho$

Fig. 2

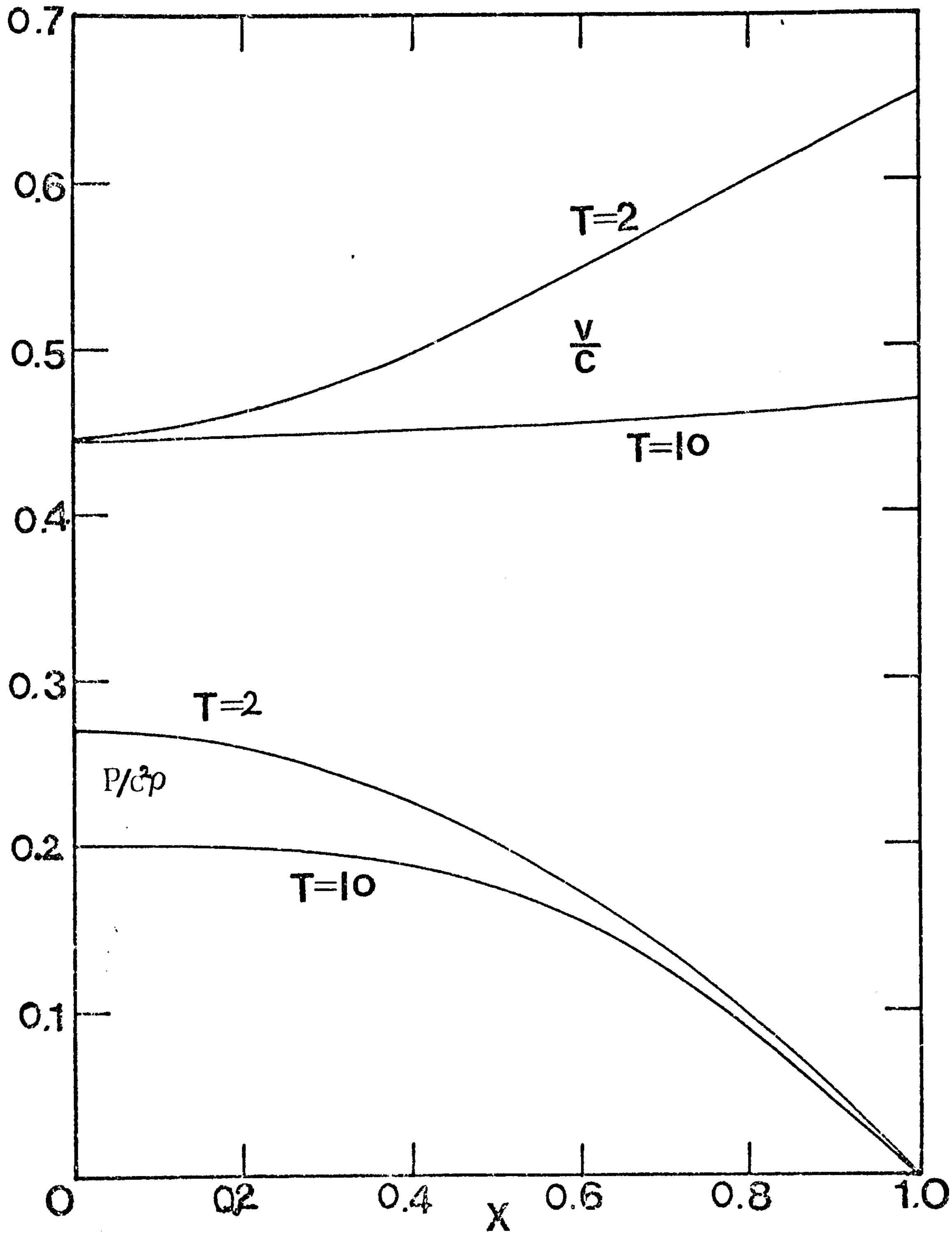

Fig. 3

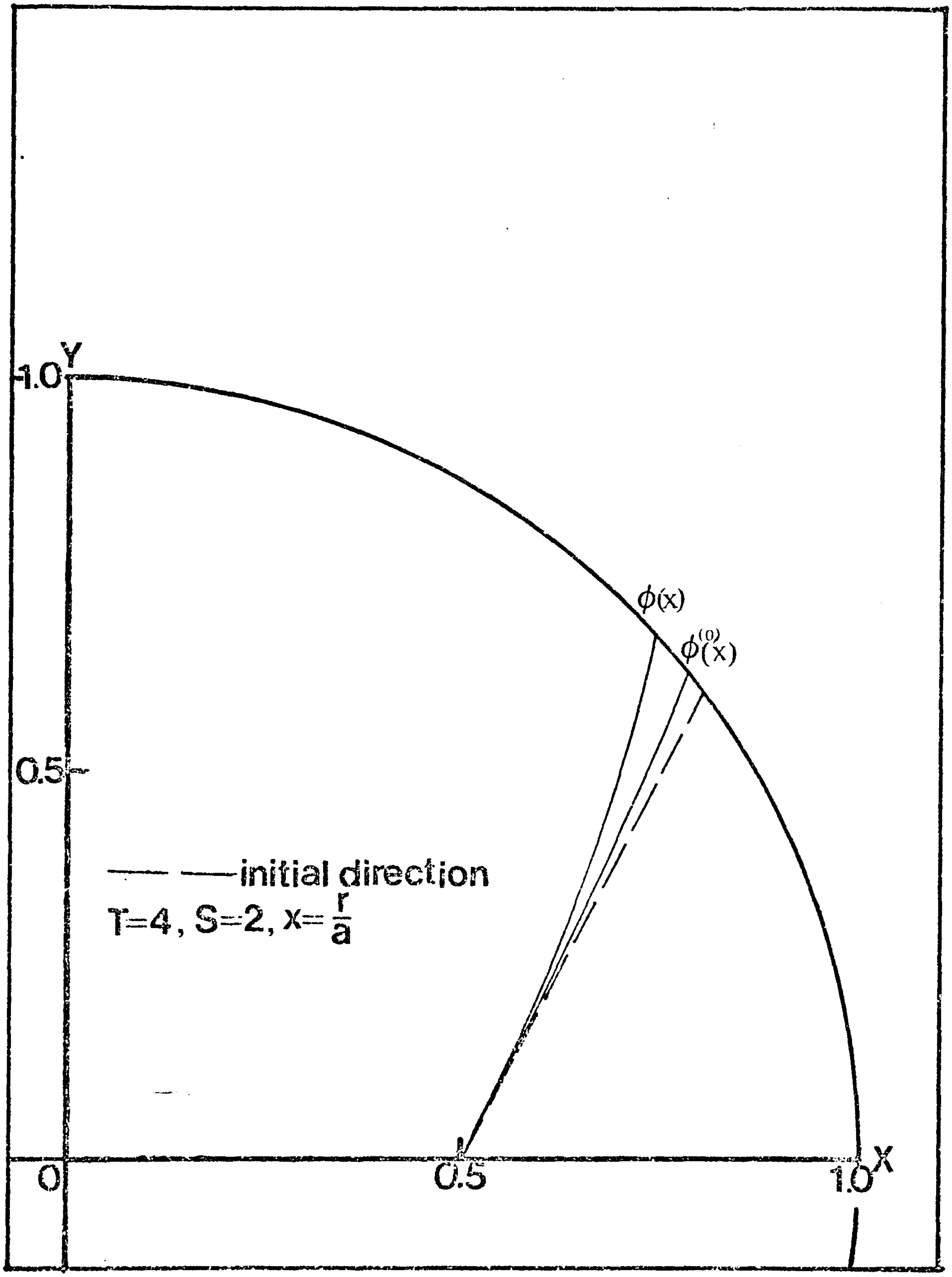

Fig. 4

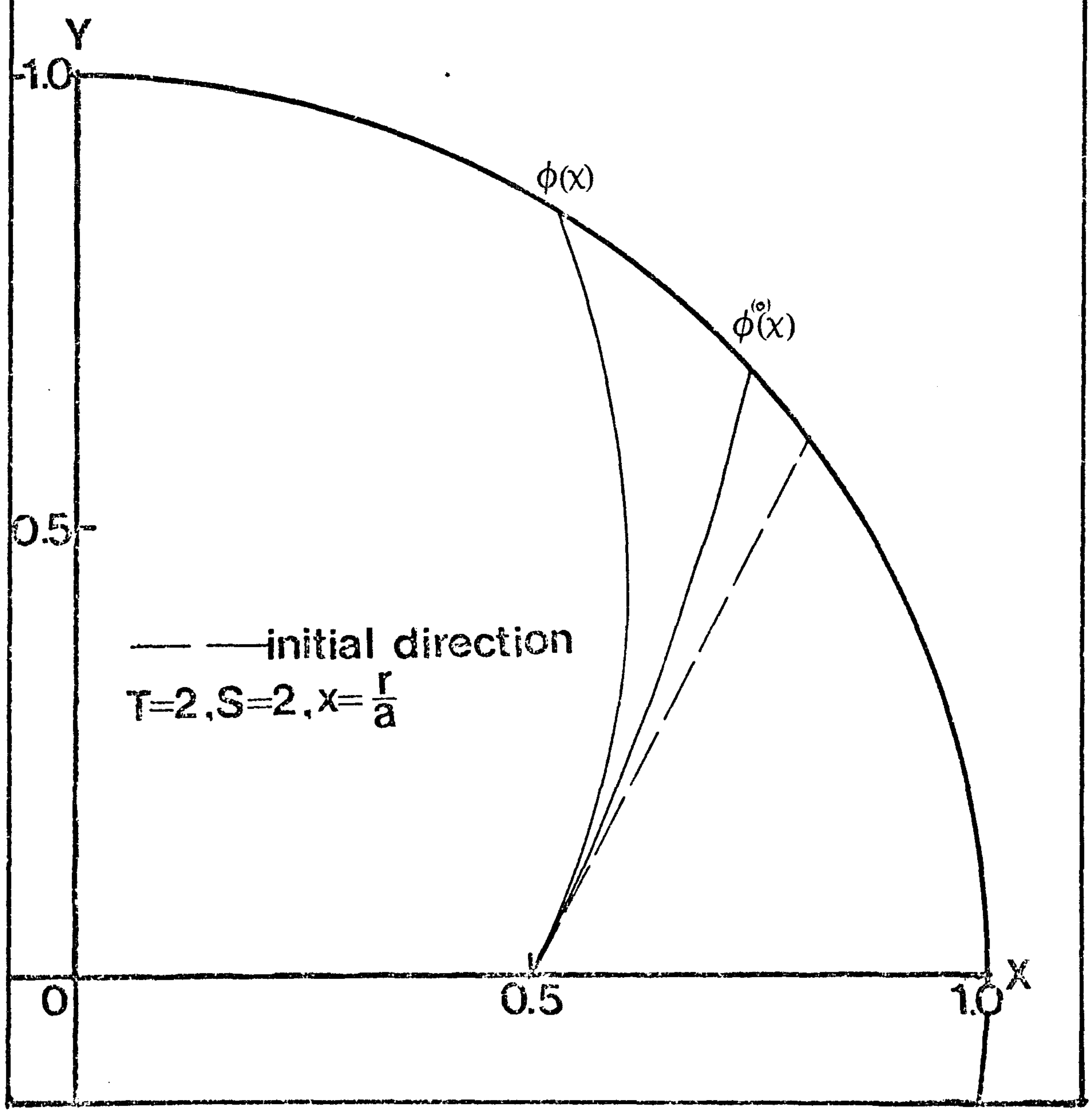

Fig. 5

VITA

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EXAMINATION AND THESIS REPORT

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Major Field: Physics

Title of Thesis: Part I: Exact Physical Properties of Electron Gases in Uniform Magnetic Fields at T = 0°K. Part II: The Path of a Sound Wave in General Relativity

Approved:

Joseph Callaway
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